

# An Equation for Generalized Variable Mass Systems and Its Consequences

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*Abstract-* In classical mechanics, Newton's second law of motion can precisely describe any nonrelativistic system, but it becomes a little confusing when the mass of a system varies throughout the time of consideration. There are many equations to deal with such systems, which are derived separately from Newton's laws. But there can be a single equation, which can deal with all variable mass systems directly. In this article, such a master equation or generalized equation has been derived and its consequences are discussed in detail.

Keywords- Variable mass system, Rocket equation, Meshchersky equation, Tsiolkovsky formula

# I. INTRODUCTION

Physicists have been dealing with variable mass systems for many decades and everything is going well. But it can be much better if there will be a single equation, which can describe any variable mass system directly. From a rocket to a raindrop to a snowball, which grows larger as it rolls downhill. There will be a single equation to deal with all such systems and that will be the master equation.

But before jumping into the derivation of such an equation, we must remember that there is no fundamental difficulty in handling any of these problems provided that we keep in mind exactly what is included in the system. According to Newton's second law,  $\vec{F} = \frac{d\vec{P}}{dt}$ , which can also be written in the integral form, as,  $\int_{t_b}^{t_a} \vec{F} dt = \vec{P}(t_a) - \vec{P}(t_b)$ . Here  $\vec{F}$  is the force on a system,  $\vec{P}(t_b)$  is the momentum of the system at the time  $t_b$  and  $\vec{P}(t_a)$  is the momentum of the system at the time  $t_a$ . While using this formula, we must deal with the same number of particles inside the system throughout the time interval  $t_a$  to  $t_b$ . Consequently, the mass of the system cannot change during the time of interest [1].

# **II. DERIVATION OF THE MASTER EQUATION**

To derive the master equation, consider a general system, which is losing and gaining some mass simultaneously with change in time. Now Newton's laws can be applied to such a system if we consider a time interval in which the mass of the system remains constant. Figure 1 below shows such a system for a very short time interval.



Figure 1 : A general variable mass system

In figure 1 at time *t*, the mass of the object is  $M + \Delta m^s$  and its velocity with respect to an inertial frame S is  $\vec{v}$ . The object can be anything, depending upon the system one considers. Here, in general a random moving object is considered, whose mass is changing with time. Also, there is an external force acting on the system. Now after a very short time,  $\Delta t$  mass of the object becomes  $M + \Delta m_a$  and its velocity relative to frame S becomes  $\vec{v} + \Delta \vec{v}$ , due to the following three reasons:

- 1. In that time interval, external force on the system accelerates the object, and hence its velocity changes.
- 2. A little amount of mass  $\Delta m^s$  is subtracted from the object in the process of losing mass, which changes its momentum and also velocity. Note that the velocity with which masses are leaving the object is  $\vec{u}^s$  and it is relative to the object itself.

3. Similarly, a little amount of mass  $\Delta m^a$  is added to the object in the process of gaining mass, which also changes its velocity and the relative velocity of masses coming towards an object is  $\vec{u}^a$ .

Now all these three factors are responsible to change an object's velocity in the time interval t to  $t + \Delta t$ . But our goal is to write an equation of motion for such a system and it can be done, if we first write the total momentum of the system at times t and  $t + \Delta t$ , then total change in momentum per unit time, to get the net external force on the system and that will be the equation of motion of our system. So, now the initial momentum of the system is,

$$\vec{P}_i = (M + \Delta m^s)\vec{v} + \Delta m^a \vec{u}^a \tag{1}$$

But, here  $\vec{v}$  is the velocity of the object relative to frame S and  $u^a$  is the velocity of the mass  $\Delta m^a$  relative to the object itself. So, in equation 1, there are velocities relative to, two frames of reference and it's not an easy thing to deal with. Therefore, we should remain only in one frame to write an equation. So, from the concept of relative velocities we have,

$$\vec{v}_{mass,object} = \vec{v}_{mass,S\,frame} - \vec{v}_{object,S\,frame}$$
 (2)

or, 
$$\vec{u}^a = \vec{v}_{mass,S\,frame} - \vec{v}$$
 (3)

or, 
$$v_{mass,S\,frame} = \vec{v} + \vec{u}^a$$
 (4)

Now put the value of equation 4 in equation 1, to get the momentum of the system with respect to frame S only. i.e.,

$$\vec{P}_i = (M + \Delta m^s)\vec{v} + \Delta m^a (\vec{u}^a + \vec{v})$$
(5)

Similarly, the final momentum of the system with respect to frame S (Like velocity  $\vec{u}^a$ , similar treatment has also been done for velocity  $\vec{u}^s$ ) is,

$$\vec{P}_f = (M + \Delta m^a)(\vec{v} + \Delta \vec{v}) + \Delta m^s (\vec{u}^s + \vec{v} + \Delta \vec{v} \quad (6)$$

And now the total change in momentum is,

$$\Delta \vec{P} = \vec{P}_f - \vec{P}_i = M \,\Delta \vec{v} + \Delta m^s \,\vec{u}^s - \Delta m^a \,\vec{u}^a \tag{7}$$

In the above equation terms like  $\Delta m^s \Delta v$  and  $\Delta m^a \Delta v$  are ignored because they become negligibly small when  $\Delta t \rightarrow 0$ .

From Newton's second law,

$$\vec{F}_{ext} = \frac{d\vec{P}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{P}}{\Delta t}$$
(8)

So,

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{P}}{\Delta t} = \lim_{\Delta t \to 0} \left[ M \frac{\Delta \vec{v}}{\Delta t} + \vec{u}^s \frac{\Delta m^s}{\Delta t} - \vec{u}^a \frac{\Delta m^a}{\Delta t} \right] \tag{9}$$

$$\vec{F}_{ext} = M \frac{d\vec{v}}{dt} + \vec{u}^s \frac{dm^s}{dt} - \vec{u}^a \frac{dm^a}{dt}$$
(10)

Now for simplicity let's write  $\frac{dm^s}{dt}$  as  $\mu^s$  and  $\frac{dm^a}{dt}$  as  $\mu^a$ . Then equation 10 becomes,

$$\vec{F}_{ext} = M \frac{d\vec{v}}{dt} + \vec{u}^s \mu^s - \vec{u}^a \mu^a \tag{11}$$

Equation 11 is the required equation of motion for the system shown in figure 1 and it is kind of a master equation for variable mass systems. But we can generalize it further to the case in which more than one mass subtracting or adding agents are present in the object [2]. In such a case, contributions from different mass-changing agents are just added together in equation 11. i.e.,

$$\vec{F}_{ext} = M(t) \, \frac{d\vec{v}}{dt} + \sum_i \vec{u}_i^s \, \mu_i^s - \sum_i \vec{u}_i^a \, \mu_i^a \tag{12}$$

Equation 12 represents the "**master equation**" for variable mass systems. Here M(t) is the mass of the object at some time t,  $\vec{u}_i^s$  and  $\vec{u}_i^a$  are velocities with which  $i^{th}$  agent subtracts or adds mass to the object relative to the object itself. Similarly,  $\mu_i^s$  and  $\mu_i^a$  are the rates with which  $i^{th}$  agent subtracts or adds mass to the object respectively. Now let's see different situations in which the master equation can be applied directly without applying Newton's laws separately and will see some famous equations popping out from the master equation.

### **III. ROCKETS**

A rocket in its simplest form is a chamber enclosing a gas under pressure. A small opening at one end of the chamber allows the gas to escape and in doing so provides a thrust that propels the rocket in opposite direction. With space rockets, the gas is produced by burning propellants that can be solid or liquid in form or a combination of both.

Now from the master equation, we can easily derive the equation of motion for a simple space rocket. Suppose at a certain time t mass of a rocket moving with velocity  $\vec{V}$  relative to the ground is M. Inside its engine some kind of fuel is burning and is coming out as gas with a velocity  $\vec{u}$  relative to the rocket itself. This exhaust velocity  $\vec{u}$  is determined by the nature of propellants, the throttling of the engine, etc., but it is independent of the velocity of the rocket [1].

Again, let's assume that the rate at which the fuel is burning in the rocket is  $\frac{dm}{dt} = \mu$ , where *m* is the instantaneous mass of fuel and it is obvious that  $\frac{dm}{dt} = -\frac{dM}{dt}$ . Because  $\frac{dm}{dt}$  is the rate of increase of exhaust mass and this mass comes from the rocket itself.

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Figure 2 : A rocket in a gravitational field g

Now for equation 12, we have the object's velocity  $\vec{v}$  as  $\vec{V}$ , mass M,  $\mu^s = \frac{dm}{dt}$ ,  $\vec{u}^s = \vec{u}$  and  $\vec{u}^a = 0$  as there is only one mass subtracting agent and no mass adding agent present in the rocket. So finally, equation 12 becomes,

$$\vec{F}_{ext} = M \frac{d\vec{v}}{dt} + \vec{u} \frac{dm}{dt}$$
(13)

$$\vec{F}_{ext} = M \frac{d\vec{v}}{dt} - \vec{u} \frac{dM}{dt}$$
(14)

Equation 14 is known as the "Fundamental Rocket Equation" or the "Meshchersky equation". The solution of this equation, assuming a constant  $\vec{u}$  and  $\frac{dM}{dt}$ , and no external force, is the "Tsiolkovsky Formula" for the final speed of a rocket.

i.e.,  $\vec{F}_{thrust} = M \frac{d\vec{v}}{dt} = \vec{u} \frac{dM}{dt}$ and,

$$\vec{V}_f = -\vec{u} \, \ln \frac{M_0}{M} \tag{15}$$

Equation 15 is called as the Tsiolkovsky Formula. Where,  $\vec{V}_f$  is the final or instantaneous velocity of the rocket at time t,  $M_0$  is the initial mass of the rocket and M is the instantaneous mass of the rocket. But the situation is quite different if we turn on a gravitational field (supplies External force) around the rocket. The equation of motion for such a system becomes,

$$M\vec{g} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt}$$
(16)

In this equation, we can also consider frictional forces or drag forces provided by atmospheric air. But it can be neglected because its effects will be less than the gravitational force. For example, a rocket of mass 12,000 kg, moving in earth's gravitational field experiences a drag force, which is only 2 percent of the gravitational force [3]. So, by solving equation 16 we get,

$$\vec{V}_f = -\vec{u} \, \ln\frac{M_0}{M} + \vec{g}t \tag{17}$$

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Notice that in equation 15, the final velocity is independent of how mass is released. The fuel can be expended rapidly or slowly without affecting the final velocity. But now in equation 17 notice that there is an advantage of burning fuel rapidly. The shorter the burn time, the greater the velocity. This is why the take-off of a large rocket is so spectacular. It is essential to burn the fuel as quickly as possible [1].

## **IV. A HYPOTHETICAL SITUATION**

Now let's consider a hypothetical situation as shown in figure 3 below.



Figure 3 : A hypothetical situation

The situation is the same as the previous one, but at an instant suddenly a source appears above the rocket, which can add mass to it and the rate at which this source is adding mass, is the same as the rate at which fuel is burning in the rocket. Means,  $\frac{dm^a}{dt} = \frac{dm^s}{dt}$  and let's assume that  $\vec{u}^a = \vec{u}^s$ . Then equation 11 becomes,

$$\vec{F}_{ext} = M \frac{d\vec{v}}{dt}$$
 or,  $M\vec{g} = M \frac{d\vec{v}}{dt}$  (18)

This means that the rocket won't get any thrust at all and soon it will begin to fall under gravity. Many more such cases can be considered by changing the various parameters of a rocket, but the master equation can handle it all without any difficulty. Now let's see our last situation, in which mass will be only deposited on an object, and how the master equation handles it.

#### V. THE FALLING RAINDROP

Consider a raindrop falling vertically under gravity, which gathers moisture from the atmosphere at a rate given by  $\frac{dm}{dt} = kt^2$ . Where *m* is the instantaneous mass of raindrop, *t* is time and  $k (= 12 \frac{gm}{s^3})$  is a constant. Suppose the drop starts falling at t = 0, with zero initial velocity and initial mass  $m_0 = 2 gm$ . We need to find its velocity as a function of time, to see how it moves through the atmosphere. Assume that the acceleration due to gravity is  $g = 1000 \ cm/s^s$ .

Now from the master equation, we have,

$$\vec{F}_{ext} = M \frac{d\vec{v}}{dt} - \vec{u}^a \mu^a \tag{19}$$

All other terms are ignored because, there is only one source, which can add mass to our object raindrop. Furthermore, it is given that,  $\mu^a = \frac{dm^a}{dt} = \frac{dm}{dt} = kt^2$ . In equation 19,  $\vec{u}^a$  is the relative velocity with which moisture is absorbed to raindrop with respect to raindrop itself and it is -v, where v is the velocity of raindrop at some time t. So now equation 19 becomes,

$$m\vec{g} = M\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}$$
(20)

If we consider downward direction positive then,

$$mg = m\frac{dv}{dt} + v\frac{dm}{dt}$$
(21)

or, 
$$g = \frac{dv}{dt} + v \frac{kt^2}{m}$$
 (22)

Now by solving this ordinary differential equation and after applying proper boundary conditions, we get

$$v = g \left[ \frac{t + \frac{t^4}{2}}{1 + 2t^3} \right] \tag{23}$$

This is the required expression for the velocity of the raindrop as a function of time, obtained from our master equation. Also, the velocity vs time plot for the raindrop is shown in the figure below. In the plot, notice that, the velocity first increases linearly, then becomes constant for a while and increases linearly again. But why such behaviour? In the absence of any kind of mass adding agent, there would have only a single straight line, as the slope of the line or the acceleration(g) is a constant. But in this case, a mass adding agent is present and the acceleration changes with change in time. The constant velocity portion of the plot, refers to the time when,  $g = v \frac{kt^2}{m}$  and net acceleration,  $\frac{dv}{dt} = 0$ . So, at that time, the velocity becomes constant and then increases, with further increase in time.



#### **VI. CONCLUSION**

So far, we have seen several examples, where the master equation was able to describe them without any complexity. In all those cases one could have considered a small time interval in which the mass of systems remained constant and then could have applied Newton's laws to them separately. But it would be very time-consuming and boring as well. On the other hand, the master equation can solve any variable mass system problem within seconds, because all the previous prescriptions of considering time intervals then applying Newton's laws, etc. have already been done in its derivation and it is just ready to be used in a proper situation.

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