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Reissner-Nordström blackholes from the induced gravity of a binary action

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Abstract— Just a few years back, it has been shown by R. A. El-Nabulsi that a simple modified gravity model based on the occurrence of the two independent actions divulge some properties that bear a resemblance to induced gravity and their corresponding cosmological and astrophysical models. Here we elaborate on his idea and propose a precise Reissner-Nordström type solution corresponding to the field equations of induced gravity from this binary action. This solution describes the gravitational field in the vicinity of a charged, non-rotating, spherically-symmetric blackhole. Furthermore, various applications are made by deriving the corresponding singularities, null geodesic equation and the orbit of photons and charged particles.

Keywords— Induced gravity, Reissner-Nordström solution.

I. INTRODUCTION

Einstein's General Theory is characterized by the existence of an invariant line element given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

Where $g_{\mu\nu}$ is the metric tensor in an arbitrary coordinate system x^μ. The metric tensor is governed by the field equation obtained from

$$\frac{1}{2\kappa}\delta \int R\sqrt{-g}\, d^4x = 0 \tag{1.2}$$

which is

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.3}$$

where $T_{\mu\nu}$ is the energy density tensor and

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{1.4}$$

Using the above Einstein's field equation, one can obtain the value of the metric tensor $g_{\mu\nu}$ which then determines how physical quantities and gravitational field varies around a mass.

The first solution, known as Schwarzschild's solution, derived by Karl Schwarzschild in 1916 describes an ideal non-rotating, uncharged blackhole. Another famous solution known as Reissner-Nordstrom solution derived by Hans Reissner [1], Hermann Weyl [2], Gunnar Nordström [3] and George Barker Jeffery [4] independently between 1916 to 1921 described the geometry outside a charged, non-rotating, spherically symmetric blackhole of mass M [5]:

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{1.5}$$

where

$$\Delta = 1 - \frac{2GM}{r} + \frac{G}{r^2}(Q^2 + P^2) \tag{1.6}$$

The above metric is obtained by letting

$$T_{\mu\nu} = F_{\mu\rho} F^{\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{1.7}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor and solving it for the equation

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.8}$$

where $R_{\mu\nu}$ is the Ricci tensor.

The aim of this paper is to provide a Reissner-Nordström type solution to the field equations of induced gravity from a binary action. This theory of gravity was originally introduced in [6] and subsequent applications were made to cosmology and the corresponding Schwarzschild type solution was also derived. The structure of binary action will be of the form (2.1). This action was also used to shade some light on the relation between classical oscillators and quantum damped oscillators [7,8]. More literature on induced gravity can be found in the references [10]-[14].

The proposed method for deriving the solution is simple and conventional. First, we assign a value to energy momentum tensor and then rewrite the modified Einstein's field equation in the form that would correspond to the zero trace of the energy momentum tensor. We then list the non-zero values of components of Ricci tensor and energy momentum tensor and then substitute their values in the modified Einstein's field equation which altogether in the end would result in a differential equation that would be solved and then the values of metric tensor will be derived. This would result in a modified Reissner-Nordström solution which would describe a charged, non-rotating, spherically-symmetric blackhole. Subsequent applications will be made to determine the motion of photons and charged particles around this blackhole by solving the geodesics equation.

The paper is organized as follows. In section 2, we provide a brief overview of modified Einstein's field equation from [6] that we would subsequently be using throughout the paper. In section 3, we derive the corresponding Reissner-Nordström solution. And finally, section 4, 5, and 6 are devoted in providing the applications such as deriving the null geodesic equation, circular orbit of photons and motion of charged particles respectively. Later section is devoted in some remarks and conclusions.

II. RELATED WORK

In what follows from the work of author from [6] is to construct a binary action of the form

$$S = S_1 + \log(e^{aS_2} - b), \quad a, b \in \mathbb{R} \tag{2.1}$$

and derive the corresponding field equations from $\delta S = 0$. The particular choice of S_1 and S_2 that we are going to employ is

$$S_1 = \int \sqrt{-g}\, L_m\, d^4x, \tag{2.2}$$

and

$$S_2 = \int \frac{1}{2\kappa}\sqrt{-g}R d^4x - \lambda(x)\left(\sqrt{-g}R - \epsilon\right). \tag{2.3}$$

Here, $\lambda(x)$ is the Lagrange multiplier introduced by forcing the constraint $(-g)^{1/2} R = \in$ where $\in$ is any real or imaginary constant. The constraint results in an action which is not invariant under full diffeomorphism. Varying action 2.1 with respect to the metric tensor yields the following modified Einstein's field equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{2a\kappa' e^{a\epsilon V_4}}{e^{a\epsilon V_4} - b} T_{\mu\nu} \tag{2.4}$$

where κ' is such that $1/\kappa' = 1/\kappa - 2\lambda$ and V_4 is the proper volume defined by

$$V_4 = \int \sqrt{-g}\, d^4x. \tag{2.5}$$

In the presence of cosmological constant, the above equation takes the form

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{2a\kappa'^{\,e^{a\epsilon V_4}}}{e^{a\epsilon V_4} - b} T_{\mu\nu}. \tag{2.6}$$

Since energy momentum tensor has zero trace, we can write Eqn. (2.4) as

$$R_{\mu\nu} = \frac{2a\kappa' e^{a\epsilon V_4}}{e^{a\epsilon V_4} - b} T_{\mu\nu}. \tag{2.7}$$

III. REISSNER-NORDSTRÖM SOLUTION

We now derive the corresponding Reissner-Nordström solution of the Eqn. (2.7). Since calculating the components of the Einstein's field equation can be a painstaking task, we are going to readily provide the values of Ricci tensor and energy momentum tensor.

We know that if

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{3.1}$$

and

$$F_{\mu\nu} = \begin{pmatrix} 0 & f(r,t) & 0 & 0 \\ -f(r,t) & 0 & 0 & 0 \\ 0 & 0 & 0 & g(r,t)r^2\sin\theta \\ 0 & 0 & -g(r,t)r^2\sin\theta & 0 \end{pmatrix} \quad (3.2)$$

then the components of the Ricci tensor and energy momentum tensor can be given as [5]

$$\begin{aligned} R_{tt} &= [\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}] + e^{2(\alpha-\beta)}\left[\alpha'' + (\alpha')^2 - \alpha'\beta' + \frac{2}{r}\alpha'\right] \\ R_{rr} &= -\left[\alpha'' + (\alpha')^2 - \alpha'\beta' + \frac{2}{r}\alpha'\right] + e^{2(\beta-\alpha)}[\ddot{\beta} + \dot{\beta}^2 - \dot{\alpha}\dot{\beta}] \\ R_{tr} &= \frac{2}{r}\dot{\beta} \\ R_{\theta\theta} &= e^{-2\beta}[r(\beta' - \alpha') - 1] + 1 \\ R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} T_{tt} &= \frac{f^2}{2}e^{-2\beta} + \frac{g^2}{2}e^{-2\alpha} \\ T_{rr} &= -\frac{f^2}{2}e^{-2\alpha} - \frac{g^2}{2}e^{2\beta} \\ T_{tr} &= 0 \\ T_{\theta\theta} &= \frac{r^2g^2}{2} + \frac{r^2f^2}{2}e^{-2(\alpha+\beta)} \\ T_{\phi\phi} &= T_{\theta\theta}\sin^2\theta \end{aligned} \quad (3.4)$$

respectively. Here, dot denotes differentiation with respect to time “t” and prime denotes differentiation with respect to “r”. Furthermore, α and β are such that $\alpha + \beta = 0$. Using Maxwell’s equations

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}T_{\mu\nu}\right) \quad (3.5)$$

and

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0 \quad (3.6)$$

we can find that

$$f(r,t) = f(r) = \frac{Q}{\sqrt{4\pi}r^2} \quad (3.7)$$

and

$$g(r,t) = \frac{P}{\sqrt{4\pi}r^2} \quad (3.8)$$

where Q and P are total electric and magnetic charges respectively. Now, consider the $\theta\theta$ component of Eqn. (2.7):

$$R_{\theta\theta} = \frac{2a\kappa' e^{a\epsilon V_4}}{e^{a\epsilon V_4} - b}T_{\theta\theta}. \quad (3.9)$$

Using Eqn. (3.3) and (3.4), we get

$$e^{-2\beta}[r(\beta' - \alpha') - 1] + 1 = \frac{2a\kappa' e^{a\epsilon V_4}}{e^{a\epsilon V_4} - b}\left[\frac{r^2g^2}{2} + \frac{r^2f^2}{2}e^{-2(\alpha+\beta)}\right] \quad (3.10)$$

$$-e^{2\alpha}[2r\alpha' + 1] + 1 = \frac{2a\kappa' e^{a\epsilon V_4}}{e^{a\epsilon V_4} - b}\left[\frac{r^2}{2}(g^2 + f^2)\right] \quad (3.11)$$

$$e^{2\alpha}[2r\alpha' + 1] = 1 - \frac{2a\kappa' e^{a\epsilon V_4}}{e^{a\epsilon V_4} - b}\left[\frac{1}{8\pi r^2}(P^2 + Q^2)\right]. \quad (3.12)$$

Since $(re^{2\alpha})' = e^{2\alpha}[2r\alpha' + 1]$, the integral

$$\int d(re^{2\alpha}) = \int \left[1 - \frac{2a\kappa' e^{a\epsilon V_4}}{e^{a\epsilon V_4} - b}\left[\frac{1}{8\pi r^2}(P^2 + Q^2)\right]\right] dr \tag{3.13}$$

yields

$$e^{2\alpha} = 1 + \frac{R_s}{r} + \frac{1}{8\pi r^2}\frac{2a\kappa' e^{a\epsilon V_4}}{e^{a\epsilon V_4} - b}(P^2 + Q^2) \tag{3.14}$$

Since in the absence of charge, the above quantity should reduce to Schwarzschild's solution, then this allows us to take the constant R_s to be $R_s = 2GM$. Thus, substituting the above value in Eqn. (3.14) and using the fact that $\alpha + \beta = 0$, we get the following modified version of Reissner-Nordström metric:

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{3.15}$$

were

$$\Delta = 1 - \frac{2GM}{r} + \frac{1}{8\pi r^2}\frac{2a\kappa'^{e^{a\epsilon V_4}}}{e^{a\epsilon V_4} - b}(P^2 + Q^2). \tag{3.16}$$

This solution will describe the gravitational field around a charged, non-rotating, spherically-symmetric blackhole of an induced gravity. We can further compute the value of V_4 as [6]

$$V_4 = \int \sqrt{-g}d^4x = \int_0^r 4\pi r^2 \int_0^t dt = 32\pi r^3 t/3. \tag{3.17}$$

Thus, we can rewrite Δ as

$$\Delta = 1 - \frac{2GM}{r} + \frac{1}{8\pi r^2}\frac{2a\kappa' e^{\xi}}{e^{\xi} - b}(P^2 + Q^2). \tag{3.18}$$

Where $\xi = 32a\epsilon\pi r^3 t/3$. Since V_4 depends on the physics it describes, we would prefer to not assign a particular value and thus write $\xi = a\epsilon V_4$ Throughout any application presented next, assume that P = 0. If dt is an infinitesimal time interval measured by an observer at an infinite distance from the gravitational field and $d\tau$ is an infinitesimal time interval measured by an observer at a distance r, then

$$d\tau = dt\sqrt{1 - \frac{2GM}{r} + \frac{Q^2}{8\pi r^2}\frac{2a\kappa' e^{\xi}}{e^{\xi} - b}} \tag{3.19}$$

Suppose that an electromagnetic wave is propagating radially from a point (r_1, θ, ϕ) at t_1 to (r_2, θ, ϕ) at t_2. Let t'_1 be the coordinate time when the electromagnetic wave completes its one oscillation period after the first emission from r_1 and t'_2 be the coordinate time when the electromagnetic wave has completed one oscillation period after receiving the signal at r_2. If $\Delta t_2 = t'_2 - t_2 = \Delta t_1 = t'_1 - t_1$, then

$$\Delta\tau_1 = \Delta t_1\sqrt{1 - \frac{2GM}{r_1} + \frac{Q^2}{8\pi r_1^2}\frac{2a\kappa' e^{\xi}}{e^{\xi} - b}} \tag{3.20}$$

and

$$\Delta\tau_2 = \Delta t_2\sqrt{1 - \frac{2GM}{r_2} + \frac{Q^2}{8\pi r_2^2}\frac{2a\kappa' e^{\xi}}{e^{\xi} - b}}. \tag{3.21}$$

Thus, we can write the proper frequency as

$$f_2 = f_1\left(1 - \frac{2GM}{r_2} + \frac{Q^2}{8\pi r_2^2}\frac{2a\kappa' e^{\xi}}{e^{\xi} - b}\right)^{1/2}\left(1 - \frac{2GM}{r_1} + \frac{Q^2}{8\pi r_1^2}\frac{2a\kappa' e^{\xi}}{e^{\xi} - b}\right)^{-1/2}. \tag{3.22}$$

A quicker look at the derived modified metric suggests that event horizon should occur when $\Delta = 0$. This yields the following quadratic equation

$$r^2 - r_s r + r_Q^2 = 0 \tag{3.23}$$

where r_s=2GM/r and

$$r_Q = \frac{Q^2}{8\pi r}\frac{2a\kappa'^{e^{a\epsilon V_4}}}{e^{a\epsilon V_4} - b}. \tag{3.24}$$

For the quadratic equation (3.23), we get the following solution:

$$r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4r_Q^2}\right). \tag{3.25}$$

In the case of $r_s > 2r_Q$, there are two coordinate singularities at r_+ and r_- and another obvious singularity at r = 0 In this case, the metric can be divided into 3 regions: $r_+ < r < r_\infty$, $r_- < r < r_+$ and $0 < r < r_-$. In the unphysical case of $r_s < 2r_Q$, there is no singularity for r > 0. The singularity at r = 0 still exists. And finally, when $r_s = 2r_Q$, their exists only one horizon located at $r = r_s/2$ and Δ is positive on both sides on the horizon. The singularity r = 0 also exists in this case.

IV. Null Geodesic and Motion of Photons

We now take a look at those equations that describes the motion of non-charged particles and photons in the modified Reissner-Nordström metric. Let $x^\alpha = x^\alpha(\lambda)$ be a curve parametrized by λ. We know that geodesic equation for x^α is

$$\frac{d^2x^\alpha}{d^2\lambda} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{4.1}$$

Solving the above equation for α = 0, 1, 2, 3 yields

$$\frac{d^2t}{d\lambda^2} + \frac{\Delta'}{\Delta}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0, \tag{4.2}$$

$$\frac{d^2r}{d\lambda^2} + \frac{\Delta'}{2\Delta^{-1}}\left(\frac{dt}{d\lambda}\right)^2 + \frac{\Delta^{-1\prime}}{2\Delta^{-1}}\left(\frac{dr}{d\lambda}\right)^2 - \frac{r\sin^2\theta}{B}\left(\frac{d\phi}{d\lambda}\right)^2 = 0, \tag{4.3}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0, \tag{4.4}$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} - 2\cot\theta\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} = 0, \tag{4.5}$$

respectively. Using above equations and some routine calculations, one can arrive at [9]:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4e^2}{L^2} - r^2\Delta\left(1 + \frac{r^2e_0^2}{L^2}\right) \tag{4.6}$$

where

$$e_0^2 = -\Delta^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{e^2}{\Delta} - \frac{L^2}{r^2}, \tag{4.7}$$

$$e = \frac{dt}{d\lambda}\Delta \tag{4.8}$$

and L is the angular momentum per unit mass. Putting the value of Δ for P = 0 yields

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4e^2}{L^2} - r^2 + 2GMr - \frac{Q^2}{8\pi}\frac{2a\kappa' e^\xi}{e^\xi - b} - \frac{r^4e_0^2}{L^2} + \frac{2GMr^3e_0^2}{L^2} - \frac{r^2e_0^2Q^2}{8\pi L^2}\frac{2a\kappa' e^\xi}{e^\xi - b} \tag{4.9}$$

e_0 must be equal to unity for time like geodesics and zero for null geodesics. Thus, we have

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4e^2}{L^2} - r^2 + 2GMr - \frac{Q^2}{8\pi}\frac{2a\kappa' e^\xi}{e^\xi - b} - \frac{r^4}{L^2} + \frac{2GMr^3}{L^2} - \frac{r^2Q^2}{8\pi L^2}\frac{2a\kappa' e^\xi}{e^\xi - b} \tag{4.10}$$

for time like geodesics and

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4e^2}{L^2} - r^2 + 2GMr - \frac{Q^2}{8\pi}\frac{2a\kappa' e^\xi}{e^\xi - b} \tag{4.11}$$

for null geodesics. A massive non-charged particle will follow a time-like geodesic while photon will follow a null geodesic.

V. Motion of Charged Particles

The Lagrangian for charged particle is

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + qA_\mu\dot{x}^\mu \tag{5.1}$$

where q is the charge per unit mass of the particle, A_μ is the four potential, a dot represents differentiation with respect to proper time. Simplifying further, we can write Lagrangian as

$$\mathcal{L} = \frac{1}{2}\left[\Delta\dot{t}^2 + \Delta^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right] + \frac{qQ'\dot{t}}{r} \tag{5.2}$$

where $Q'=Q/4\pi\mathcal{E}_0$. Using Euler's-Lagrange equation, we get [9]

$$\left(\frac{dr}{d\phi}\right)^2 = -r^2\Delta\left(1+\frac{r^2}{L^2}\right) + \frac{r^4}{L^2}\left(e-\frac{qQ'}{r}\right)^2. \tag{5.3}$$

Substituting the value of Δ with P = 0 yields

$$\left(\frac{dr}{d\phi}\right)^2 = -r^2 + 2GMr\left(1+\frac{r^2}{L^2}\right) - \frac{2a\kappa' e^{\xi}}{e^{\xi}-b}\left(\frac{Q^2}{8\pi}+\frac{r^2Q^2}{8\pi L^2}\right) + \frac{r^4}{L^2}\left(\left(e-\frac{qQ'}{r}\right)^2 - 1\right). \tag{5.4}$$

This is the equation of motion that governs the motion of a charged particle in the modified Reissner-Nordström metric. In the case when q = 0, Eqn. (5.4) reduces to an equation which describes a free falling non-charged massive particle.

VI. FUTURE SCOPE

In our derived metric (3.15), when we let P = Q = 0, we get the standard Schwarzschild's solution corresponding to Einstein's gravity equations. One might alter the case and expect the solution to reduce at modified Schwarzschild's solution [6]. This means the value of Δ metric (3.15) should have the form

$$\Delta = 1 - \frac{M\kappa'}{4\pi(1-be^{-a\epsilon V_4})r} + \frac{1}{8\pi r^2}\frac{2a\kappa'^{e^{a\epsilon V_4}}}{e^{a\epsilon V_4}-b}(P^2+Q^2) \tag{6.1}$$

We have presented the above assumption because the metric with above value of Δ is difficult to derive from the induced gravity equations. The corresponding physical quantities such as time dilation, singularities, motion of photon and charged particles can be easily derived after putting (6.1) in the metric (3.15) and using derived geodesic equations. We hypothesize that a similar approach can be done towards deriving the other solutions of the induced gravity field equations such as Kerr solution and Kerr-Newman solution.

VII. CONCLUSION

In summary, we have derived a Reissner-Nordström solution for induced gravity equations corresponding to a binary action. This solution would describe the vicinity of a charged, non-rotating, spherically-symmetric blackhole of an induced gravity that corresponds to a binary action. With the extra terms that are involved in the derived metric, it will provide a more accurate insight than the standard Reissner-Nordström solution corresponding to the standard Einstein's field equation. Furthermore, we have calculated the related singularities, path of photos and motion of charged particles near this charged blackhole. Our solution, in the absence of magnetic and electric fields yields the standard Schwarzschild's solution as expected. Further applications of the above presented solution are in progress and will be presented in another paper.

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