Construction of Optimal Pairwise Balanced Block Designs Based on $3^n$ Symmetrical Factorial Design

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Available online at: www.isroset.org

Received: 02/Jun/2019, Accepted: 22/Jun/2019, Online: 30/Jun/2019

Abstract—A new method for constructing Optimal Pairwise Balanced Block Design using the concept of $3^n$ symmetrical factorial design has been proposed. The constructed method is illustrated with numerical example and the design is found to be universal optimal.

Keywords—Block design, Incidence matrix, Balanced Design, Pairwise Balanced Block Designs, Concurrent matrix

I. INTRODUCTION

Let us consider a class of block designs in which v treatments are arranged in b blocks according to the replication number of the $i^{th}$ treatment, $r_i = \sum_{j=1}^{b} n_{ij}$ (replication number of the $i^{th}$ treatment) and $k_j = \sum_{i=1}^{v} n_{ij}$ (the $j^{th}$ block size).

Bose and Shrikhande [1] discussed about the Pairwise Balanced Block Design (PBBD) in addition to existence of orthogonal Latin square designs. Bose and Shrikhande [2] obtained the various methods for the construction of Pairwise orthogonal sets of Latin square design. The concept of PBBD is merely the combinatorial interest in block designs. Because with the help of PBBD many other incomplete block designs can be constructed.

Bose and Shrikhande [3] defined PBBD as :

Definition: An arrangement of v treatments in b blocks is defined as PBBD of index $\lambda$ of type $(v; k_1, k_2, ..., k_m)$ provided

a) Each set contains $(k_1, k_2, ..., k_m)$ symbols that are all distinct.

b) $k_i \leq v; k_i \neq k_j$ and

c) every pair of distinct treatments occurs in exactly $\lambda^2$ sets of the design.

Further they also showed some of the parametric relation of PBBD which is namely:

$$b = \sum_{i=1}^{m} b_i \text{ and } \lambda(v-1) = \sum_{i=1}^{m} b_i(k_i-1)$$

(1)

A characterization of PBBD in terms of concurrent matrix $NN^T$ can be expressed in the following way:

$$\left(r-\lambda\right) I_v + \lambda E_{vv}$$

(2)

where $I_v$ is a identity matrix of order $(v \times v)$ and $E_{vv}$ is the unit matrix of order $(v \times v)$

The above equation can be rewritten as

$$NN^T = \text{Diag}(r-\lambda) + \lambda E_{vv}$$

(3)

In this paper, section one contains the introduction, definition and some parametric conditions related to PBBD, section two contains the review of related works, section three contains the procedure of constructing the PBBD and section four contains the conclusion.

II. RELATED WORK

Design (PBIBD) and also obtained optimality. Rajarathinam et al., [6], constructed PBBD using 2^k factorial design. Das [5] studied optimality criteria and derived some results on optimal block designs. In this paper, the optimal criteria of A, D and E optimal block design have been checked.

In this paper a new method of constructing PBBD based on the concept of 3^n symmetrical factorial design is proposed and testes for optimality criteria of the design are carried out.

III. METHODOLOGY

The procedures for the construction of unequal block sizes, PBBD using the concept of 3^n symmetrical factorial experiment is discussed below.

A. Construction of Pairwise Balanced Block Design by deleting control treatment and merging all the main effects

A lemma without proof for the construction of ternary PBBD of unequal block sizes and equi-replicated design is given below.

**Lemma 1:** Consider a 3^n symmetrical factorial experiment.

(i) From these 3^n treatment combinations delete the control treatment. i.e., delete (0,0,0, …, 0).

(ii) Merge all those treatment combinations which represent n main effects and further consider these n merged treatment combinations as one treatment combinations. This way, we have 3^n - (2n-1) treatment combinations.

(iii) Finally, consider 3^n - (2n-1) treatment combinations as the block for the required design.

For an example, let n=2, the 3^2=9 treatment combinations are

\[
\begin{array}{cccccccccc}
0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 2 \\
\end{array}
\]

Delete control treatment. Merge all treatment combinations where the level of one factor is two, while the level of other factors are zero, and next merge all the treatment combinations whose level of one factor is one, while the level of other factors are zero. Keep the remaining treatment combinations as such.

Finally, the treatment combinations are

\[
\begin{array}{cccc}
2 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 \\
2 & 2 & 2 & 2 \\
\end{array}
\]

Call this matrix N which is the combinations of 3^n - (2n-1) treatment combinations. Finally the matrix (transpose of N) becomes the incidence matrix of ternary PBBD.

**Theorem 1:** There always exists a series of non-proper, PBBD with parameters

\[ v = n, b = 3^n - (2n-1), r = 3^n, \]

\[ k = [21^c_1 c_2,31^c_2 c_3,\ldots,(2n-b)1^c_n1^c_1,2n1^c_1], \]

\[ \lambda_1 = \lambda_3 = 3^{n-2} + 1, \lambda_2 = 2 \times 3^{n-2} \]

whose incidence matrix N is given by Lemma 1.

**Proof:** Consider a 3^n symmetrical factorial experiment. This has 3^n treatment combinations. Considering n factors as rows and 3^n treatment combinations as columns and then using Lemma 1 one will have the following incidence matrix of a design d.

\[ N = \left[\begin{array}{cccc}
0 & 1 & 2 & \cdots & 2 & 2 \\
1 & 0 & 1 & \cdots & 2 & 2 \\
1 & 1 & 0 & \cdots & 2 & 2 \\
0 & 0 & 0 & \cdots & 2 & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & 2 \\
\end{array}\right] \]

Since there are n rows and considering these as treatments, obviously the number of treatments is \(v = n\).

By the above lemma, by deleting the control treatment and merging all main effects treatment combinations finally one can get \(3^n - (2n-1)\) treatment combinations. Since these are treated as blocks, number of blocks of the design is \(b = 3^n - (2n-1)\).

Evidently, number of unity elements in columns of N is as follows:

2 with frequency \(^nC_2\), 3 with frequency \(2n (^nC_n)\), etc.,

(2n-1) with frequency \(^nC_1\) and 2n with frequency \(^2C_1\). This gives \(k = [21^c_1 c_2,31^c_2 c_3,\ldots,(2n-1)1^c_n1^c_1,2n1^c_1] \).

Since in each row 1 and 2 occurs \(3^n\) times, so the number of replication is \(r = 3^n\). Using the incidence matrix N shown in (4) the following C-matrix is obtained.
\[ C = R - NK^{-1}N^T \]
\[
\begin{bmatrix}
\alpha & \beta & \beta & \cdots & \beta \\
\beta & \alpha & \beta & \cdots & \beta \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta & \beta & \beta & \cdots & \alpha
\end{bmatrix}
= (\alpha - \beta) I_v + \beta E_{vv}
\] (5)

where \( \alpha = r_i - \sum \frac{n_{ij}^2}{k_j} \) and \( \beta = -\sum \frac{n_{ij}n_{jj}}{k_j} \).

The non-zero eigen values of the above C-matrix is
\[ \theta = \alpha \left[ \frac{v}{v-1} \right] \]
with multiplicity \( v-1 \) and zero with multiplicity 1, where \( v \) is the number of treatments.

For the design having the incidence matrix \( N \) given in equation (4), the concurrent matrix \( NN^T \) is given below.
\[
NN^T = (5 \times 3^{n-1} - 3^n - 5) I_v + (3^n + 5) E_{vv}
\]
\[
NN^T = \begin{bmatrix}
5 \times 3^{n-1} & 3^n + 5 & \cdots & 3^n + 5 \\
3^n + 5 & 5 \times 3^{n-1} & \cdots & 3^n + 5 \\
\vdots & \vdots & \ddots & \vdots \\
3^n + 5 & 3^n + 5 & \cdots & 5 \times 3^{n-1}
\end{bmatrix}
\] (9)

In the above matrix all the off diagonal elements are same and it can be expressed as
\[
NN^T = \left( \frac{5}{3} \times r - 9 \lambda_1 - 5 \right) I_v + (9 \lambda_1 + 5) E_{vv}
\] (10)

Hence it proves that the incidence matrix \( N \) given in (4) gives improper, equi-replicated, PBBD with parameters,
\[
v = n, \ b = 3^n - (2n - 1), \ r = 3^n,
\]
\[
k = [21_{n-2}, 31_{(n-1)}, \ldots, (2n-1)1_{(n-2)}, 2n1_{(n-1)}]
\]
\[
\lambda_1 = \lambda_3 = 3^{n-2} + 1, \ \lambda_2 = 2 \times 3^{n-2}
\]

1. Numerical illustrations
Consider \( 3^3 \) symmetrical factorial designs. The incidence matrix using the Lemma 1 is as follows.
\[
N = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 2 & 1 & 2 & 0 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 0 & 2 & 2 & 1 & 2 & 2 & 2
\end{bmatrix}
\]
\[
N^T = \begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 2 & 2 & 0 & 1 & 2 & 1 & 2 & 2 & 2
\end{bmatrix}
\] (11)
The C-matrix of a block design is obtained as
\[
C = \frac{461}{60} \begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}
\] (12)
The above C-matrix can be further simplified as
\[
C = \frac{461}{20} \left[ I_3 - \frac{1}{3} E_{33} \right] = \frac{461}{20} I_3 - \frac{461}{60} E_{33}
\] (13)

The characteristics roots are \( \frac{461}{20} \) and \( -\frac{461}{60} \) with multiplicities 2 and 1 respectively. The \( NN^T \) is
\[
NN^T = \begin{bmatrix}
135 & 86 & 86 & 86 \\
86 & 135 & 86 & 86 \\
86 & 86 & 135 & 86 \\
86 & 86 & 86 & 135
\end{bmatrix}_{4 \times 4}
\]

Obviously this matrix satisfies the conditions of PBBD. Hence the resulting design is a PBBD with parameters
\[
v = 3, \ b = 22, \ r = 27,
\]
\[
k = (2 \times 3^1, 3 \times 3^1, 4 \times 3^1, 51^1, 61^1),
\]
\[
\lambda_1 = \lambda_3 = 10, \lambda_2 = 18
\]

B. Optimality

Das [5] studied the optimality criteria and some results on optimal block designs. In this paper, optimal criteria of A, D and E are used.

Let \( d \) belongs \( D(v, b, r, k, \lambda) \) with \( C_d \) matrix, where
\[
C = R - NK^{-1}N^T
\] Design \( d \) will be A - Optimal if it maximizes \( tr(C_d) \).
\[
tr(C_d) = tr(R) - tr(N_{kk}^T N_{kj}) = tr(R) - tr(N_{kk}^T N_{kj})
\] (15)

For a design \( d \), it can be shown that the sum of the variances of the estimates of all elementary treatment contrast is proportional to the sum of the reciprocals of the non-zero eigen values of \( C \). Thus, a design which is \( A \)-Optimal for inferring on a full set of orthonormalized treatment contrast is optimal for the estimation of the overall elementary treatment contrast.

Let \( \theta_1, \theta_2, \theta_3, \ldots, \theta_{v-1} \) are non-zero eigen values. As we know that, for variance balanced design there will be only one non-zero eigen values with multiplicities \( (v-1) \) of \( C_d \) matrix of design \( d \). That is \( \theta_1 = \theta_2 = \ldots = \theta_{(v-1)} = \theta \) as \( C \) - matrix is positive semi-definite. Finally we can say that the design is \( A \)-Optimal if
\[
\sum_{i=1}^{(v-1)} \frac{1}{\theta_i} \geq \frac{(v-1)^2}{tr(C_d)}
\] (16)
D-Optimality
Let \( \theta_1, \theta_2, \theta_3, \ldots, \theta_{(v-1)} \) are non-zero eigen values with multiplicities \((v-1)\) of \( C_d \) matrix of design \( d \). A design is D-Optimal if
\[
\prod_{i=1}^{(v-1)} \frac{1}{\theta_i} \leq \prod_{i=1}^{(v-1)} \left( \frac{\sum_{i=1}^{(v-1)} \frac{1}{\theta_i}}{(v-1)} \right)
\]
(17)

E-Optimality
Let \( \theta_1, \theta_2, \theta_3, \ldots, \theta_{(v-1)} \) are non-zero eigen values with multiplicities \((v-1)\) of \( C_d \) matrix of design \( d \). A design is E-Optimal if
\[
\text{Min}\left(\theta_i\right) \leq \frac{\text{tr}(C_d)}{(v-1)}
\]
(18)

To prove optimality
Consider the PBBD obtained in numerical example 1 with parameters
\( v = 3, b = 22, r = 27, k = (21^{1}, 31^{1}, 41^{1}, 51^{1}, 61^{1}) \), \( \lambda_1 = \lambda_2 = 4, \lambda_3 = 6 \). The trace of \( C \)-matrix of PBBD comes out as 461/10 and non-zero eigen value of \( C \)-matrix is \( \theta = \frac{461}{20} \) with multiplicity 2.

Here the inequality
\[
\sum_{i=1}^{(v-1)} \frac{1}{\theta_i} \geq \frac{(v-1)^2}{\text{tr}(C_d)}
\]
\[
\left( \frac{20}{461} + \frac{20}{461} + \frac{20}{461} \right) \geq \frac{40}{461}
\]
\[
\frac{60}{461} > \frac{40}{461}
\]
holds true which is required condition for \( d \) to be A-Optimal. Hence PBBD constructed in the numerical example 1 is an A-Optimal.

Again the inequality
\[
\prod_{i=1}^{(v-1)} \frac{1}{\theta_i} \leq \prod_{i=1}^{(v-1)} \left( \frac{1}{\theta_i} \right)
\]
\[
\left( \frac{20}{461} \times \frac{20}{461} \times \frac{20}{461} \right) \leq \frac{1}{2} \left[ \frac{20}{461} + \frac{20}{461} + \frac{20}{461} \right]
\]
\[
\frac{8000}{97972181} < \frac{27000}{97972181}
\]
holds true and hence the PBBD constructed through numerical example 1 is D-Optimal.

Also the inequality
\[
\text{Min}\left(\theta_i\right) \leq \frac{\text{tr}(C_d)}{(v-1)}
\]
\[
\frac{461}{20} \leq \frac{461}{20}
\]
(21)
holds true and hence the PBBD given in numerical example 1 is E-Optimal. Since the constructed PBBD is A-Optimal, D-Optimal as well as E-Optimal, the design is universal optimal.

IV. CONCLUSION
The designs constructed using the proposed new procedure is satisfied A-Optimality, D-Optimality and E-Optimality criteria’s and hence the design constructed using the proposed procedures is found to be universal optimum.

ACKNOWLEDGMENT
Authors are thankfully acknowledged the editor and anonymous reviewer for their valuable suggestions and comments to improve the earlier version of this paper.

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