International Journal of Scientific Research in Mathematical and Statistical Sciences
Volume-7, Issue-6, pp.46-50, December (2020)

Research Paper

E-ISSN: 2348-4519

A Generalization to $\int e^x \left(f(x) + f'(x)\right)dx$

Toyesh Prakash Sharma

St. C.F Andrews School, Agra, India

Author's Mail Id: toyeshprakash@gmail.com, Tel.: 9412330448

Available online at: www.isroset.org

Received: 10/Dec/2020, Accepted: 20/Dec/2020, Online: 31/Dec/2020

***Abstract*—**. With the help of this paper, the author is providing a generalized expression of well-known integral i.e. integral of e^x(f(x)+f'(x))dx, for concluding the main result author used concept of integral by parts, the concept of mathematical induction, differentiation, integration, etc. by using author's generalization we can easily solve many time-consuming integrals which may a good thing for solvers to find the given integrals in less duration of time respectively. Generally, peoples are not focusing on finding generalizations to given integral as a problem but, yes, they can solve given integral, some of the persons try to find generalizations of given integral but due to very long procedure, they drop their idea.

Keywords—Integral by parts, Mathematical induction, differentian, Integration, exponentials etc

I. INTRODUCTION

As Integration is a different type of concept which deals with area, volume, and other concepts that arise by combining infinitesimal data, integration have lots of applications in our daily life which becomes integration helpful for us. Integral calculus has its own properties and formulas, substitution as Euler's substitution, Weierstrass substitution etc. on behalf of them there are lots of special cases and special integrals. Now, in this paper we will focus on a special case which can be obtained by Integral by parts [1][2]: -

$$\int e^x \left(f(x) + f'(x)\right)dx \tag{1.1}$$

Statement of integral by parts:[3][4] -
"The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Mathematically: -
$$\int f(x)\cdot g(x)dx = f(x)\int g(x)dx - \int \left(f'(x)\int g(x)dx\right)dx \tag{1.2}$$

This is the basic knowledge that we use further whiling discussing results.

II. OVERVIEW

We can divide this paper on the basis of different sections are as follows: -

Abstract and keywords

1. **Introduction**: in this section author briefly discuss meaning of the title and basic knowledge required
2. **Overview**: in this section we can observe content of the paper.
3. **Related Work**: this section deals with the published work related to the main title.
4. **Main Result**: this section is the main section which carried new theorems, work etc.
5. **Some Remarkable examples**: this section deals with well know problems and solution in short manner with apply introduced method
6. **Open problem:** in this section there are some open problems for renders to found and further discussion
7. **Conclusion**: this section deals with uses of introduced method, further scope of the method, and summary of the work.

Acknowledgement
References
Authors Profile

III. RELATED WORK

There are many problems related to eq. (1.1) [5]. in different books as Integral Calculus by Amit M Agrawal, Mathematics class 12 by R.D. Sharma, Mathematics of class 12 by R.S. Aggarwal etc. As till eq. (1.1) gives preference everywhere.

IV. MAIN RESULTS

We will look toward generalization by proving some new theorems and after it by using normal observation and concept of induction, stated main theorem i.e., generalization of eq. (1.1).

Theorem 1.

$$\int e^x\left(f(x)-f''(x)\right)dx=e^x\left(f(x)-f'(x)\right)+c$$

Proof:

$$I=\int e^x\left(f(x)+f''(x)\right)dx=\overbrace{\int e^x f(x)dx}^{I(1)}+\int e^x f''(x)dx$$

Now, applying integration by parts in I (1) then,

$$\overbrace{\int e^x f(x)dx}^{I(1)}=f(x)\int e^x dx-\int\left(\frac{d}{dx}f(x)\int e^x dx\right)dx$$

$$=e^x f(x)-\overbrace{\int e^x f'(x)dx}^{I(1.2)}$$

Now,

$$\overbrace{\int e^x f'(x)dx}^{I(1.2)}=f'(x)\int e^x dx-\int\left(\frac{d}{dx}f'(x)\int e^x dx\right)dx$$

$$=e^x f'(x)-\int e^x f''(x)dx$$

Then, I (1)

$$\overbrace{\int e^x f(x)dx}^{I(1)}=e^x f(x)-\overbrace{\int e^x f'^{(x)}dx}^{I(1.2)}=e^x f(x)-\left(e^x f'(x)-\int e^x f''(x)dx\right)=$$

$$e^x f(x)-e^x f'(x)+\int e^x f''(x)dx$$

So, I become as-

$$I=\int e^x\left(f(x)+f''(x)\right)dx=\overbrace{\int e^x f(x)dx}^{I(1)}+\int e^x f''(x)dx$$

$$=e^x f(x)-e^x f'(x)+\int e^x f''(x)dx+\int e^x f''(x)dx$$

$$=e^x f(x)-e^x f'(x)+2\int e^x f''(x)dx$$

Then,

$$I=\int e^x\left(f(x)+f''(x)\right)dx=e^x f(x)-e^x f'(x)+2\int e^x f''(x)dx$$

$$\int e^x\left(f(x)+f''(x)\right)dx-2\int e^x f''(x)dx=e^x f(x)-e^x f'(x)$$

$$\int e^x\left(f(x)-f''(x)\right)dx=e^x\left(f(x)-f'(x)\right)+c$$

Hence prove.

Theorem 2.

$$\int e^x\left(f(x)+f'''(x)\right)dx=e^x\left(f(x)-f'(x)+f''(x)\right)+c$$

Proof

As,

$$I = \int e^x \left(f(x) + f'''(x)\right)dx = \overbrace{\int e^x f(x)dx}^{I(1)} + \int e^x f'''(x)dx$$

By applying integral by parts in I(1)

$$\overbrace{\int e^x f(x)dx}^{I(1)} = f(x)\int e^x\, dx - \int \left(\frac{d}{dx}f(x)\cdot \int e^x\, dx\right)dx$$

$$= e^x\, f(x) - \overbrace{\int e^x f'(x)\, dx}^{I(2)} + C_1$$

By applying integral by parts in I(2)

$$\overbrace{\int e^x f'(x)\, dx}^{I(2)} = f'(x)\int e^x\, dx - \int \left(\frac{d}{dx}f'(x)\cdot \int e^x\, dx\right)dx$$

$$= e^x f'(x) - \overbrace{\int e^x f''(x)dx}^{I(3)} + C_2$$

By applying integral by parts in I(3)

$$\overbrace{\int e^x f''(x)dx}^{I(3)} = f''(x)\int e^x\, dx - \int \left(\frac{d}{dx}f''(x)\cdot \int e^x\, dx\right)dx$$

$$= e^x f''(x) - \overbrace{\int e^x f'''(x)dx}^{I(4)} + C_3$$

Now,

$I = I(1) + I(4) = e^x\, f(x) - I(2) + I(4) = e^x\, f(x) - \left(e^x f'(x) - I(3)\right) + I(4) =$
$e^x\, f(x) - e^x f'(x) + I(3) + I(4) = e^x\, f(x) - e^x f'(x) + e^x f''(x) - I(4) + I(4) =$
$I = e^x\left(f(x) - f'(x) + f''(x)\right) + C_1 + C_2 + C_3$

Hence prove.

Theorem 3.

$$\int e^x \left(f(x) - f''''(x)\right)dx = e^x\left(f(x) - f'(x) + f''(x) - f'''(x)\right) + C_2$$

Proof

By the same procedure theorem 1 can also written as-

$$\int e^x \left(f''(x) - f''''(x)\right)dx = e^x\left(f''(x) - f'''(x)\right) + c$$

Let consider it as eq (ii)

$$\int e^x \left(f''(x) - f''''(x)\right)dx = e^x\left(f''(x) - f'''(x)\right) + c_1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (i)$$

And consider above theorem be eq (i)

$$\int e^x \left(f(x) - f''(x)\right)dx = e^x\left(f(x) - f'(x)\right) + c \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (ii)$$

By adding both the equation we have

$$\int e^x \left(f(x) - f''(x)\right)dx + \int e^x \left(f''(x) - f''''(x)\right)dx = e^x\left(f(x) - f'(x)\right) + e^x\left(f''(x) - f'''(x)\right) + (c_1 + c\,)$$

$$\int e^x \left(f(x) - f''''(x)\right)dx = e^x\left(f(x) - f'(x) + f''(x) - f'''(x)\right) + C_2$$

Here, C_2 denoted $(c_1 + c\,)$

So, following the same way we can also prove the following identities: -

$$\int e^x \left(f(x) + f'(x)\right)dx = e^x f(x) + c$$
$$\int e^x \left(f(x) - f''(x)\right)dx = e^x\left(f(x) - f'(x)\right) + c$$

$$\int e^x \left(f(x)+f'''(x)\right)dx = e^x\left(f(x)-f'(x)+f''(x)\right)+c$$
$$\int e^x \left(f(x)-f''''(x)\right)dx = e^x\left(f(x)-f'(x)+f''(x)-f'''(x)\right)+c$$
$$\int e^x \left(f(x)+f'''''(x)\right)dx = e^x\left(f(x)-f'(x)+f''(x)-f'''(x)+f''''(x)\right)+c$$
$$\int e^x \left(f(x)-f''''''(x)\right)dx = e^x\left(f(x)-f'(x)+f''(x)-f'''(x)+f''''(x)-f'''''(x)\right)+c$$
$$\int e^x \left(f(x)+f'''''''(x)\right)dx = e^x\left(f(x)-f'(x)+f''(x)-f'''(x)+f''''(x)-f'''''(x)+f''''''(x)\right)+c$$
$$\int e^x \left(f(x)-f''''''''(x)\right)dx = e^x\left(f(x)-f'(x)+f''(x)-f'''(x)+f''''(x)-f'''''(x)+f''''''(x)-f'''''''(x)\right)+c$$

Now by observing above identities and by induction [6] we are presenting main generalization i.e.

Theorem 4: *if* $n \in 1,2,3,4,\cdots$ *then,*

$$\int e^x\left(f(x)-(-1)^n\frac{d^n}{dx^n}f(x)\right)dx = e^x\left(\sum_{k=1}^{n}(-1)^{(k-1)}\frac{d^{(k-1)}}{dx^{(k-1)}}f(x)\right)+C$$

V. SOME REMARKABLE EXAMPLES

Some examples related to above theorem are as follows: -

1. $\int e^x \sin x\, dx$[7]

Sol.

As

$$\int e^x \left(f(x)-f''(x)\right)dx = e^x\left(f(x)-f'(x)\right)+c$$

Let

$f(x)=\sin x \Rightarrow f'(x)=\cos x \Rightarrow f''(x)=-\sin x$

Then,

$$\int e^x\left(\sin x-(-\sin x)\right)dx = e^x(\sin x-\cos x)+c$$
$$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x-\cos x)+c$$

2. $\int e^x \cos x\, dx$[8]

Sol.

As

$$\int e^x \left(f(x)-f''(x)\right)dx = e^x\left(f(x)-f'(x)\right)+c$$

Let

$f(x)=\cos x \Rightarrow f'(x)=-\sin x \Rightarrow f''(x)=-\cos x$

Then,

$$\int e^x\left(\cos x-(-\cos x)\right)dx = e^x(\cos x+\sin x)+c$$
$$\int e^x \cos x\, dx = \frac{e^x}{2}(\cos x+\sin x)+c$$

3. $\int e^x\left(\ln x+\frac{1}{x^2}\right)dx$

Sol.

As

$$\int e^x \left(f(x)-f''(x)\right)dx = e^x\left(f(x)-f'(x)\right)+c$$

Let

$f(x)=\ln x \Rightarrow f'^{(x)}=\frac{1}{x} \Rightarrow f''(x)=-\frac{1}{x^2}$

Then,

$$\int e^x\left(\ln x-\left(-\frac{1}{x^2}\right)\right)dx = e^x\left(\ln x-\frac{1}{x}\right)+c$$
$$\int e^x\left(\ln x+\frac{1}{x^2}\right)dx = e^x\left(\ln x-\frac{1}{x}\right)+c$$

- We can solve these problems by applying integral by parts two time but with applying given theorem 1 we can easily evaluate given integral.
- With the help of here introduced method we can save our time.
- Yes. The need to solve given integral is to found f(x).

VI. OPEN PROBLEMS

Some open problems are as follows: -

1. Prove the main generalization without using induction.
2. Does there possible some other generalizations for $\int e^x\left(f(x)+f'(x)\right)dx$?
3. Does main generalization can connect with Taylor series?

VII. CONCLUSION AND FUTURE SCOPE

In this paper there is a generalization to the well know form i.e.

$$\int e^x\left(f(x)+f'(x)\right)dx$$

Have been proposed with the help of notable theorems, concept of induction (Mathematical Induction), General Observation and problems. After it author asks to some problems in the section open problem.

Further we can find many other applications of provided generalization, we can also find some other generalizations to this eq. (1.1).

ACKNOWLEDGMENT

Author is thankful to his Mathematics teacher Er. Manish Gupta, because from a long time on regular basis author presents his work for comment and corrections to Him, As, now it is on front of every one due to blessings of his Teachers, Parents, cousins, etc that's why author is thankful to them with involving referees and

REFERENCES

[1] Amit M Agrawal, " Integral Calulus" Ch-1 Indefinate integral session 4 integral by parts publisher- Arhint p.22 ISBN-978-93-13191-91-9.

[2] NCERT- Mathematics Textbook for class XII Ch-7 integrals Ex-7.6. *P.326* **.** ISBN-81-7450-653-5

[3] NCERT- Mathematics Textbook for class XII Ch-7 integrals Ex-7.6. *P.323-328***.** ISBN-81-7450-653-5

[4] Joseph Edwards: Integral Calculus for beginners, Ch-IV " Integral by Parts.. Publisher-arihant *P 32* ISBN-978-93-5094-145-4.

[5] R.D Sharma "Mathematics class-XII" Vol. 1 Ch- 19 Indefinate integral Ex-19.26, publication Danpat Pai Publications rivised edition 2020 ISBN-978-81-941926-5-7. *p.19.137-19.151*

[6] NCERT- Mathematics Textbook for class XI Ch-4 Principles of Mathematical induction. *P.86*

[7] NCERT- Mathematics Textbook for class XII Ch-7 integrals Example 21 before Ex-7.6. *P.326***.** ISBN-81-7450-653-5

[8] Socratic Q and A: "How to find integral of e^xcosxdx?.

AUTHORS PROFILE

Mr. Toyesh Prakash Sharma: Passed 10th from CBSE board in year 2019 from St. C. F Andrews School. Currently he is studying in 12th standard with the same board and school. From the month march 2020 his more than 15 papers have published in different journals in which IJMCT is one of them although his personal Interest in Mathematics. His personal address is B-509 Kalindi Vihar, Agra, India-282006

