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Multiple Intruder Locating Dominating Sets

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Abstract - Safeguarding of facilities has been at a major focus in graph theoretical research which has led to a surge of various parameters of locating dominating sets in graphs. However, in all those parameters, a single intruder is assumed to be present in a network. In this paper, we introduce Multiple Intruder Locating Dominating (*MILD*) sets where one is interested in finding the presence as well as the locations of intruders at multiple (possibly all) locations in a network. The number of vertices in the smallest *MILD* set of a graph G is called its *MILD* number, denoted by *MILD*(G). For a simple connected graph of order n, the *MILD* number lies between n/2 and n-1, both inclusive. The graphs which attain these bounds are characterized.

Keywords— *Domination, Locating Domination, Multiple intruder locating domination.*

I. INTRODUCTION

The problem of finding a smallest subset S of vertices, called a dominating set of a graph G = (V, E) where every vertex not in S is dominated by (or adjacent to) at least one vertex in S of G, has vast applications ranging from social networks to mobile networks to resource management. Hence there has been an explosion of interest in the topic leading to the emergence of research papers on many variations of the problem and books documenting them [1],[2],[4],[5],[12]. The problem of safeguarding a facility with optimal number of detectors leads to one such variation of domination called location - domination and has been studied in many papers [6],[7],[8],[10],[11]. In all those works, the presence of a single intruder in a network is considered. Suppose there are multiple intruders in a network, possibly at all locations, then we are interested in knowing precisely each of the locations where intruders are present.

To achieve this, suppose a network is represented using a graph G = (V, E). We place a detector each at a set $S \subseteq V$ of vertex locations. A vertex location with detector is called a *codeword*. Each detector at a vertex v can transmit four signals:

- 0 if no intruder is in $N[v] = N(v) \cup v$.
- 1 if an intruder is in $N(v) \cap (V \setminus S)$ and no intruder is at v.
- 2 if an intruder is at v and no intruder is in $N(v) \cap (V \setminus S)$.
- 3 if the intruders are at both v and $N(v) \cap (V \setminus S)$.

where $N(v) = \{w \in V: vw \in E\}$ is called the open neighborhood of v.

Suppose $\forall u \in V \setminus S, \exists v \in S$ such that $N(v) \cap (V \setminus S) = \{u\}$, then based on the signals sent by the detectors, intruders at any number of locations in a network can be located precisely. For example, consider the network in Figure 1. Here, $S = \{a, c, d, g\}$. Suppose intruders are at the locations a, b, c and f, then the signals received from the detectors will be (3, 3, 0, 1). By this, we can see that the intruders can be located unambiguously. Hence, a subset $S \subseteq V$ with the above said property is called a *Multiple Intruder Locating Dominating set* of G, or in short, a *MILD set* of G. The minimum cardinality of a *MILD* set in G is called the *Multiple Intruder Locating Domination number* of G, denoted by *MILD(G)* or $\gamma_{ml}(G)$.

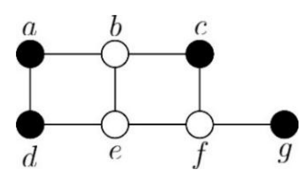
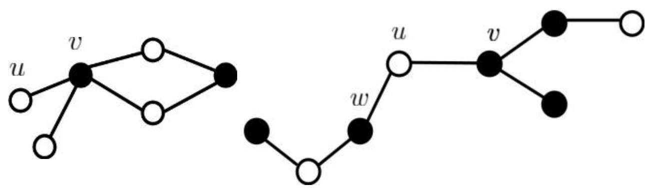


Figure 1. An example of a *MILD* set in a graph

Whenever for a vertex u in $V \setminus S, \exists v \in S$ such that $N(v) \cap (V \setminus S) = u$, we say v is the *devout dominator* of u and u is the secure non - codeword of v. A devout dominator together with its secure non - codeword is called a code pair.

The definitions of devout dominator and secure neighbor resemble with the ones of ‘sole dominator’ and ‘private neighbor’ defined by Peter J Slater [11]. The vertex v is called a *sole dominator* of u and u is called a private neighbor of v whenever $N[u] \cap S = v$. The following examples illustrate the differences.



(A) (B)

Figure 2. Sole dominator vs. Devout dominator

In the Figure 2(A), v is the sole dominator of u but not a devout dominator and u is a private neighbor but not the secure neighbor of v. In the Figure 2(B), v is the devout dominator of u and w is not, u is the secure neighbor of v and not of w. Also observe that v is not a sole dominator of u.

Throughout this paper, the graphs considered are simple, connected and of order n. The notations are used as per Teresa W. Haynes et. al. [4],[5] and D. B. West [13].

The organization of this paper is as follows. In Section 2, we device a method called *chaining scheme* to find the *MILD* number of a graph. In Section 3, we establish the bounds on MILD number of a graph, as well as, characterize the graphs which attain those bounds. The concluding remarks are given in the Section 4.

II. CHAINING SCHEME

To form a *MILD* set of the least cardinality for a graph *G*, one must form as many code - pairs as possible in it. It can be seen that this depends upon the number of *Matchings* possible in *G*, since it is a set *M* of independent edges which happen to establish a pairing of the vertices incident to each edge in *M* (refer [3]). Such a pair of adjacent vertices together with the edge connecting them is called *link* (Figure 3). No two links share a vertex. Two links are *adjacent* if at least two vertices, one from each link, are adjacent.

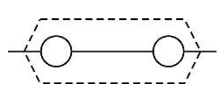


Figure 3. A Link

A vertex which could not be a part of any link is called *a lonely vertex*. Since a matching need not be unique for a graph, links can be formed in different ways without altering their overall number in the graph (Figure 4). If the matching is *perfect*, then every vertex of the graph is a part of some link.

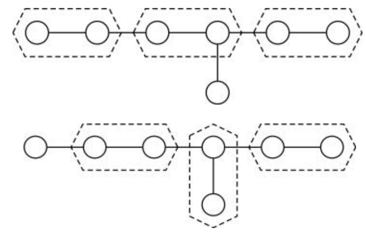


Figure 4. Different formations of links

A *chain* is a series of adjacent links where all the constituent vertices together induce a path or a cycle. The smallest chain is a single link itself. Two chains are *adjacent* if at least two vertices, one from each chain, are adjacent. Grouping the links of a graph so as to form chains is called chaining.

Consider a vertex of a chain and label it with one of the numbers ‘1’ or ‘2’. Without loss of generality, suppose the vertex is labelled ‘1’, then the other vertex in the link must be labelled ‘2’. Now in the chain, the vertex of an adjacent link that is adjacent to ‘1’ will be labelled ‘1’ and the one adjacent to ‘2’ will be labelled ‘2’, as shown in the Figure 5. This process is carried out until all the vertices in the chain are labelled. By doing the same to all chains in a graph, we get a numbering of vertices which helps in building a *MILD set* for a graph. Hence this procedure is called *MILD-numbering*. Formation of links through matching, then chaining and *MILD-numbering* altogether form a *chaining Scheme*.

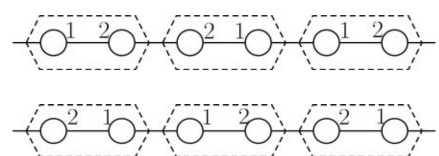


Figure 5. Two ways of *MILD*-numbering a chain

Once a chaining scheme is applied to a graph *G*, let us shade all the lonely vertices (if any) and all the vertices numbered *2* (or *1*) with a color, say black. Suppose we consider the shaded vertices as codewords, we can proceed to check if they form a *MILD set*. Since this shading of vertices helps in forming a *MILD* set of *G*, it is called a *MILD-shading*.

Now, by applying the *MILD*-shading technique, we discuss the *MILD* number of paths $\boldsymbol{P_n}$ and then, extend it to cycles $\boldsymbol{C_n}$.

Proposition 1. Consider a path $\boldsymbol{P_n}$. Perform the chaining scheme, and then, by applying a MILD-shading it can be easily seen that $\boldsymbol{\gamma_{ml}(P_n) = \lceil n/2 \rceil}$

Proposition 2. $\gamma_{ml}(C_n) = \gamma_{ml}(P_n) + c$ *where*

$$c = \begin{cases} 1, & n = 4k+2, k \in Z^+ \\ 0, & otherwise \end{cases}$$

Proof. We consider the following cases that can arise.

(i) When *MILD* shading is applied to P_{4k}, P_{4k+1} and P_{4k+3}, joining of the two end vertices will not affect the *MILD* set.

(ii) When *MILD* shading is applied to P_{4k+2}, one end vertex, say *u*, will be a devout dominator. Thus, joining *u* with the other end vertex say *v*, which will be a non-codeword, affects its devout domination. Thus, either *v*, or the secure non-codeword adjacent to *u* must also become a codeword.

This proves the result.

III. BOUNDS ON MILD NUMBER OF A GRAPH

Proposition 3. *For a graph G with* $n \geq 2$,

$$n/2 \leq \gamma_{ml}(G) \leq n - 1.$$

Proof. Every codeword can devout dominate only one non-codeword. Hence the least number of codewords in a graph will be in a case where every codeword devout dominates a non-codeword. Thus, the lower bound follows, and is attained by paths P_{2k}, cycles C_{4k} ($k \in Z^+$), etc.

Consider two vertices v and u in G. Suppose u must devout dominate v. If all other vertices (if any) in G are adjacent to u, then all those vertices must be codewords. In such a scenario, the upper bound follows and is attained by star graphs, complete graphs, etc.

A. *The Upper Bound.*

Lemma 4. *If a graph G has* P_4 *or* C_4 *as induced subgraph then* $\gamma_{ml}(G) \leq n - 2$.

Proof. If a given graph G has P_4 or C_4 as induced subgraph, then by applying chaining scheme and MILD shading for that portion of the graph, we have two secure non-codewords. By making all other vertices (if any) adjacent to the devout dominators into codewords, the result follows.

Corollary 5. *If a graph G has* P_k *or* C_k *as induced subgraph then* $\gamma_{ml}(G) \leq n - \gamma_{ml}(P_k)$ *or* $\gamma_{ml}(G) \leq n - \gamma_{ml}(C_k)$ *respectively.*

Proposition 6. *([12]) If a graph G does not have* P_4 *or* C_4 *as induced subgraph then G has a vertex adjacent to all other vertices.*

Theorem 7. *For a graph G, if* $\gamma_{ml}(G) = n - 1$ *then* $\Delta(G) = n - 1$.

Proof. Suppose $\Delta(G) \neq n - 1$ for the given graph G. By contrapositive of Proposition 3.4, G contains P_4 or C_4 as induced subgraph. Then by Lemma 3.2, $\gamma_{ml}(G) \neq n - 1$. This proves the result. However, the converse is not true. The next result proceeds on that matter.

Proposition 8. *For a graph* $G(V, E)$ *with* $n \geq 4$, *if* $\Delta(G) = n - 1$ *then* $\frac{n}{2} + 1 \leq \gamma_{ml}(G) \leq n - 1$

Proof. The upper bound is the general one established in Proposition 3.1. With $\Delta(G) = n - 1$, G can have the maximum number of secure non-codewords when the vertex/vertices of degree $n - 1$ are non devout dominating codewords, so that, when the remaining vertices (not of degree $n - 1$) induce P_{2k} or C_{4k} ($k \in N$), the lower bound would follow.

Theorem 9. *For a graph G with* $n \geq 4, \gamma_{ml}(G) \leq n - 2$ *if and only if G contains* P_4, C_4, *or butterfly-graph as induced subgraph.*

Proof. Suppose $\gamma_{ml}(G) \leq n - 2$, then there are more than one code pairs in G. Consider two code pairs with vertices, say, p1-q1 and p2-q2 and let them form links L1 and L2 respectively. Then the two links are either adjacent or non-adjacent.

They are adjacent (i.e, the distance between them is one) if and only if their vertices together induce P_4 or C_4.

If they are not adjacent, then the distance between them must be two or more. Suppose the links are at a distance two from each other, then there will be a vertex, say x, connecting the links. If only one vertex from each link is adjacent to x, then P_5 is formed which contains P_4 in itself. If two vertices of a link, say L1, and one vertex, say p2, of the other link L2 are adjacent to x, then q2, p2, x and p1 induce P_4. If both the vertices from each link are adjacent to x then butterfly graph is induced.

Suppose they are at a distance three, then there are two vertices, say x and y, on the path connecting the links. Without loss of generality, suppose p1 from L1 is adjacent to x and p2 from L2 is adjacent to y. Then p1xyp2 induce P_4. Similar argument follows if there are three vertices on the path connecting the links (and if four are there, then they induce P_4 themselves!).

Conversely, suppose P_4 or C_4 is induced, then by Lemma 3.2, there are at least two code pairs in G. In case of a butterfly graph being induced, the vertices of degree less than three can form code pairs.

The result follows.

By contrapositive of Theorem 3.7, the following result, which characterizes the graphs attaining the upper bound, immediately follows.

Corollary 10. For a graph G, $\gamma_{ml}(G) = n - 1$ if and only if G contains none of P_4, C_4, or butterfly-graph as induced subgraph.

Consider a graph G with δ(G) = 1. The vertex adjacent to a pendant vertex in G is called a support vertex.

Proposition 11. In a graph G, suppose there are q support vertices, then $\gamma_{ml}(G) \leq n - q$.

The result follows from a simple fact that every pendant vertex can devout dominate (or be devout dominated by) its support vertex.

Theorem 12. *For an $n - 2$ regular graph G with $n \geq 4, \gamma_{ml}(G) = n - 2$.*

Proof. Given G is n−2 regular which implies every vertex is not adjacent to one vertex in G. Consider a vertex x not adjacent to a vertex y. Another vertex, say z, will be adjacent to x and y but not adjacent to a vertex p. But again, p must be adjacent to x and y. Thus, xzyp is a cycle C_4. By Lemma 3.2, $\gamma_{ml}(G) \leq n - 2$.

Conversely, in the given regular graph G, if a vertex x devout dominates a vertex y, then all the other $n - 3$ vertices in N(x) must also become codewords. Thus, with x counted, there are $n - 2$ codewords and we have $\gamma_{ml}(G) \geq n - 2$. Hence the proof.

Corollary 13. *Let G be an $n - p$ regular graph with $p \leq n - 2$, then $\gamma_{ml}(G) \geq n - p$.*

Theorem 14. *For an $n - 3$ regular graph G with at least five vertices, $n - 3 \leq \gamma_{ml}(G) \leq n - 2$, and the bounds are sharp.*

Proof. Given *G* is regular and hence, every vertex is not adjacent to exactly two vertices. Consider a vertex *x*, and, let *y* be one of the two vertices that *x* is not adjacent to. Out of the $n - 3$ vertices that *y* is adjacent to, maximum one of them can be not adjacent to *x*. Consider a vertex *z* adjacent to both *x* and *y* but not adjacent to a vertex *q*. Now, *q* is not adjacent to two vertices one of them being *z*. Hence *q* should be adjacent to either *x* or *y* (or both). If *q* happens to be adjacent to *x* as well as *y*, then the vertices *x*, *y*, *z* and *q* together induce the subgraph C_4 else, they induce P_4. The upper bound follows from Lemma 3.2.

From Corollary 3.11, the lower bound follows. The sharpness of both the bounds is established by the graphs in the Figure 6.

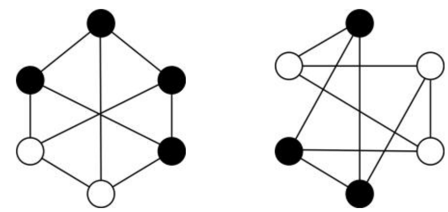


Figure 6. $n - 3$ regular graphs with *MILD* numbers $n - 2$ and $n - 3$.

B. The Lower Bound

We now characterize the graphs which attain the lower bound.

Definition. *Consider a graph G, perform chaining scheme on it. Without loss of generality, whenever any vertex labeled* 1 *of a chain c*1 *is adjacent with only, say, the vertex/ vertices numbered* 2 *in another chain c*2*, we intend to give a same color to the 1 vertices of c*1 *and the 2 vertices of c*2 *(black or white/shaded or unshaded). We represent this by writing as follows:*

1(c1) = 2(c2)

Theorem 15. *For any graph G, $\gamma_{ml}(G) = n/2$ if and only if the graph has a chaining scheme such that:*

(1) *the links form a perfect matching.*
(2) *the chaining produces r chains none of them inducing C_{4k+2}.*
(3) *for any two adjacent chains ci and cj (1 ≤ i ≤ r , 1 ≤ j ≤ r, i ≠ j) only one of the following is true:*
    (i) 1(*ci*)= 2(*cj*) *and* 1(*cj*) = 2(*ci*).
    (ii) 1(*ci*) = 1(*cj*) *and* 2(*cj*) = 2(*ci*).
(4) *for any three adjacent chains ci, cj and ck (i ≠ j ≠ k, 1 ≤ k ≤ r), ' = ' is transitive.*

Proof. Necessity: Given that the links form a perfect matching, also the chaining and *MILD*-numbering are done. So every vertex is part of a link which contains two vertices of different numbers viz. *1* and *2*. Since none of the chains induce $C_{4k+2} (k \in N)$, the adjacent vertices of any two adjacent links in any chain have a same number.

Consider a chain *ci*, let its vertices labeled *1* be colored black and the ones labeled *2* be colored white.

In (3) if (i) is true, then all vertices labeled *2* in *cj* must be colored black and those labeled *1* must be colored white. Else, the *1* vertices of *cj* must be colored black and the *2* vertices must be colored white. Thus, two adjacent vertices of any two adjacent chains *ci* and *cj* will have same color.

Consider another chain *ck* adjacent to both *ci* and *cj* satisfying (3) with both *ci* and *cj*. Then the issue that may arise is (without loss of generality): suppose 1(*ci*) = 2(*cj*), 2(*cj*) = 2(*ck*) and 1(*ci*) = 1(*ck*), then there is a conflict where both *1* vertex and *2* vertex of *ck* will have to be assigned same color. Since '=' is transitive, such a situation is averted. Thus, a vertex u of a chain, say *ci*, and the vertices of other two chains that *u* is adjacent-to can be assigned a same color.

Due to transitivity, a similar argument can be made for the cases with more than three adjacent chains. Thus, an arbitrary vertex u of an arbitrary chain *ci* is adjacent to only one vertex (in its link) of different color than itself and all other vertices (be it in the chain *ci* or any number of other chains adjacent to *ci*) adjacent to u will be of same color as u.

By the discussion until here, it can be seen that when the vertices of all the chains are colored, we end up with a black vertex and a white vertex in each of the $n/2$ links such that each vertex will be adjacent to only one vertex of the opposite color. What that means is by considering black vertices as codewords and the white ones as non-codewords, every codeword devout dominates a non-codeword. The result follows.

Further it can be observed that in the beginning, for the chain that we considered, if the *1* vertices are colored white and the *2* vertices are colored black, then we get a completely different set of codewords (which are same in the number anyway).

Sufficiency: Suppose $\gamma_{ml}(G) = n/2$. That means there are as many codewords as non-codewords, or in other words, every codeword devout dominates a non-codeword (i.e, every codeword is adjacent to only one non-codeword). Let every codeword and its secure non-codeword be grouped together to form a link. Since every vertex is thus part of one or the other link, there is a perfect matching.

Let the codewords be numbered *1* and the non-codewords *2*. Let every link be considered as a chain. Then each vertex will be adjacent to a vertex of same number outside its link and thus, outside its chain, (2) and (3)(ii) are satisfied and, the transitivity follows too. Hence the sufficiency.

IV. CONCLUSION AND FUTURE SCOPE

Multiple Intruder Locating Dominating (*MILD*) set in graphs is introduced, by which, multiple intruders in a network can be located. The bounds for *MILD* number of a graph are established and with the help of a method called chaining scheme, the graphs which attain those bounds are characterized. In future, we intend to develop algorithms to find MILD numbers of different classes of graphs.

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