On \((gg)^*\)- Closed Sets in Topological Spaces

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Available online at: www.isroset.org
Accepted 18/Aug/2018, Online 30/Aug/2018

Abstract - In this paper, we introduce a new class of closed sets called generalization of generalized star closed sets (briefly \((gg)^*\) - closed) in topological Spaces. A subset \(A\) of a topological space \((X, \tau)\) is called \((gg)^*\) - closed if \(U\) contains regular closure of \(A\) whenever \(A\) contains \(U\) and \(U\) is \((gg)^*\) - open in \((X, \tau)\). We studied the relation of this set with some of the other closed and generalized closed sets and some of the characteristics, of \((gg)^*\) - closed sets have been investigated and studied.

Keywords: \((gg)^*\) - closed set, \((gg)^*\) - open, regular closure.

AMS classification 2010: 54A05

I. INTRODUCTION


In this paper we introduce a new class of closed set called \((gg)^*\) - closed sets in Topological spaces. Section 1, gives the overall introduction to the paper, followed by section 2, where we recall some of the existing closed and open sets. Section 3, provides us with the introduction to the concept of \((gg)^*\) - closed set. In section 4, the independency of \((gg)^*\) - closed sets with some of the existing closed and generalized closed sets are studied and its outcome is shown in the form of a diagram. In section 5, some of the properties of \((gg)^*\) - closed sets are studied, analyzed and proved; which leads to section 6, the conclusion of the paper. After which, the references that were dealt with during the analyses are given at the end of the paper.

II. PRELIMINARIES

Throughout this paper \((X, \tau)\) represent the topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset \(A\) of \(X\), the closure of \(A\) and interior of \(A\) are denoted by \(cl\) (\(A\)) and \(int\) (\(A\)) respectively.

Definition 2.1 A subset \(A\) of a topological space \((X, \tau)\) is called a

1) generalized - closed set (briefly g - closed) [1] if \(cl\) (\(A\)) \(\subseteq\) \(U\) whenever \(A\) \(\subseteq\) \(U\) and \(U\) is open in \(X\).

2) regular semi open [2] if there is a regular open set \(U\) such that \(U\) \(\subseteq\) \(A\) \(\subseteq\) \(cl\) (\(U\)).

3) regular open set [3] if \(A = int\) (\(cl\) (\(A\))) and a regular closed set if \(cl\) (\(int\) (\(A\))) = \(A\).

4) generalization of generalized closed set (briefly gg-closed) [4] if \(gcl\) (\(A\)) \(\subseteq\) \(U\) whenever \(A\) \(\subseteq\) \(U\) and \(U\) is regular semi - open in \(X\).
(5) semi-open set [5] if $A \subseteq cl(int(A))$ and semi-closed set if $int(cl(A)) \subseteq A$.

(6) pre-open set [6] if $A \subseteq int(cl(A))$ and pre-closed if $cl(int(A)) \subseteq A$.

(7) semi-pre open set [7] if $A \subseteq cl(int(cl(A)))$ and semi-pre-closed if $int(cl(int(A))) \subseteq A$.

(8) $\beta$-open set [7] if $A \subseteq cl(int(cl(A)))$, whenever $A \subseteq U$ and $U$ is open in $X$.

(9) $\alpha$-open set [8] if $A \subseteq int(cl(cl(A)))$ and $\alpha$-closed set if $cl(int(cl(A))) \subseteq A$.

(10) $t$-set [9] iff $int(A) = int(cl(A))$.

(11) generalized semi-pre closed (briefly gsp-closed) [10] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(12) generalized pre-closed set (briefly gp-closed) [11] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(13) generalized semi-closed set (briefly gs-closed) [12] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(14) $\alpha$-generalized closed set (briefly $ag$-closed) [13] if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(15) regular generalized closed set (briefly rg-closed) [14] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular-open in $X$.

(16) generalized pre-regular closed set (briefly gpr-closed) [15] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular-open in $X$.

(17) generalized semi-pre regular-closed set (briefly gspr-closed) [16] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular-open in $X$.

(18) generalized star pre-closed (briefly $g^*p$-closed) [17] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $X$.

(19) weakly closed set (briefly w-closed) [18] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$.

(20) tgr-closed set [19] if $rel(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $t$-set.

(21) regular w-closed (briefly rw-closed) [20] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular semi-open in $X$.

(22) regular general $\alpha$-closed set (briefly $rg\alpha$-closed) [21] if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular $\alpha$-open in $X$.

(23) generalized $\alpha$-closed set (briefly $ga$-closed) [22] if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$.

(24) Semi-generalized closed set (briefly sg-closed) [23] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$. 
(25) R*- closed set [24] if \( rcl (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular semi-open in \( X \).

(26) R\(^g\) - closed set [25] if \( gcl (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is R\(^g\) -open in \( X \).

(27) \( \beta g^* \) - closed set [26] if \( gcl (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \beta \) - open in \( X \).

(28) r\(^g\) - closed set [27] if \( cl (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular - open in \( X \).

(29) g\(^**\) - closed set [28] if \( cl (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is g\(^*\) - open in \( X \).

(30) g\(^*\) - closed set [29] if \( cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is g - open in \( X \).

(31) generalized regular closed set (briefly gr - closed) [30] if \( rcl (A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is regular in \( X \).

The complements of the above closed sets are their open sets and vice versa.

**Definition 2.2** [19] The regular closure of a subset \( A \subseteq X \) is the set \( rcl(A) = \bigcap \{ B \subseteq X : B \text{ is regular closed and } A \subseteq B \} \)

**III. GENERALIZATION OF GENERALIZED STAR - CLOSED SETS**

**Definition 3.1** A subset \( A \) of a topological space \((X, \tau)\) is called generalization of generalized star closed sets (briefly (gg)*-closed) if \( rcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is gg - open.

**Example 3.2** Let \( X = \{a, b, c, d\} \), and \( \tau = \{ \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, X \} \)

\[ \text{gg - open} = \{ \emptyset, \{a, c, d\}, \{c, d\}, \{a, d\}, \{a, c\}, \{a, b\}, \{d\}, \{b\}, \{c\}, \{a\}, X \} \]

\[ \text{(gg)* - closed} = \{ \emptyset, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X \}. \]

**Proposition 3.3** Every regular closed set is (gg)* - closed.

**Proof:** Let \( A \) be a regular closed set in \( X \) such that \( A \subseteq U \) and \( U \) is gg - open. Then \( rcl(A) = A \).

Hence \( rcl (A) \subseteq U \). Therefore \( A \) is (gg)* - closed.

**Remark 3.4** The converse of the above proposition need not be true as shown in the following example.

**Example 3.5** Let \( X = \{a, b, c, d\} \), and \( \tau = \{ \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}, X \} \)

Then the set \( \{a, b\} \) is (gg)* - closed but not regular closed.

**Proposition 3.6**

(1) Every (gg)* - closed set is g- closed.

(2) Every (gg)* - closed set is g**-closed.

(3) Every (gg)* - closed set is gsp- closed.

(4) Every (gg)* - closed set is gp- closed.

(5) Every (gg)* - closed set is gs- closed.
(6) Every $(gg)^*$-closed set is $ag$-closed.

(7) Every $(gg)^*$-closed set is $rg$-closed.

(8) Every $(gg)^*$-closed set is $gpr$-closed.

(9) Every $(gg)^*$-closed set is $gspr$-closed.

(10) Every $(gg)^*$-closed set is $g^p$-closed.

(11) Every $(gg)^*$-closed set is $gr$-closed.

(12) Every $(gg)^*$-closed set is $gr^*$-closed.

**Proof:**

(1) Let $A$ be a $(gg)^*$-closed set in $X$. Let $U$ be an open set in $X$ such that $A \subseteq U$. Since every open set is $(gg)^*$-closed, $rcl(A) \subseteq U$. Hence $A$ is g-closed.

(2) Let $A$ be a $(gg)^*$-closed set in $X$. Let $U$ be a $g^*$-open set in $X$ such that $A \subseteq U$. Since every $g^*$-open set is $(gg)^*$-open and since $A$ is $(gg)^*$-closed, $rcl(A) \subseteq U$. Hence $A$ is $g^*$-closed.

(3) Let $A$ be a $(gg)^*$-closed set in $X$. Let $U$ be an open set in $X$ such that $A \subseteq U$. Since every open set is $(gg)^*$-closed, $rcl(A) \subseteq U$. Hence $A$ is $gs$-closed.

(4) Let $A$ be a $(gg)^*$-closed set in $X$. Let $U$ be an open set in $X$ such that $A \subseteq U$. Since every open set is $(gg)^*$-closed, $rcl(A) \subseteq U$. Hence $A$ is $gp$-closed.

(5) Let $A$ be a $(gg)^*$-closed set in $X$. Let $U$ be a regular open set in $X$ such that $A \subseteq U$. Since every regular open set is $(gg)^*$-closed, $rcl(A) \subseteq U$. Hence $A$ is $gspr$-closed.

(6) Let $A$ be a $(gg)^*$-closed set in $X$. Let $U$ be a regular open set in $X$ such that $A \subseteq U$. Since every regular open set is $(gg)^*$-closed, $rcl(A) \subseteq U$. Hence $A$ is $gr$-closed.

(7) Let $A$ be a $(gg)^*$-closed set in $X$. Let $U$ be a regular open set in $X$ such that $A \subseteq U$. Since every regular open set is $(gg)^*$-closed, $rcl(A) \subseteq U$. Hence $A$ is $g^p$-closed.

(8) Let $A$ be a $(gg)^*$-closed set in $X$. Let $U$ be a regular open set in $X$ such that $A \subseteq U$. Since every regular open set is $(gg)^*$-closed, $rcl(A) \subseteq U$. Hence $A$ is $gspr$-closed.
Let A be a (gg)*-closed set in X. Let U be an open set in X such that \( A \subseteq U \).
Since every open set is gg-open and since A is (gg)*-closed, \( rcl(A) \subseteq U \). Hence A is gr*-closed.

Remark 3.7 The converse of the above proposition need not be true as shown in the following example.

Example 3.8 Let \( X = \{a,b,c,d\} \), and \( \tau = \{\emptyset, \{c\}, \{d\}, \{c,d\}, \{a,c,d\}, X\} \). Then

1. \{b\} is g-closed but not (gg)*-closed.
2. \{b\} is g**-closed but not (gg)*-closed.
3. \{a\} is gsp-closed but not (gg)*-closed.
4. \{a\} is gp-closed but not (gg)*-closed.
5. \{d\} is gs-closed but not (gg)*-closed.
6. \{b\} is \( \alpha \)-closed but not (gg)*-closed.
7. \{a, c\} is rg-closed but not (gg)*-closed.
8. \{c, d\} is gpr-closed but not (gg)*-closed.
9. \{a,c,d\} is gspr-closed but not (gg)*-closed.
10. \{a\} is g*p-closed but not (gg)*-closed.
11. \{b\} is gr-closed but not (gg)*-closed.
12. \{b\} is gr*-closed but not (gg)*-closed.

IV. INDEPENDENCY OF (gg)*-CLOSED SETS WITH OTHER CLOSED SETS.

The following example shows that (gg)*-closed sets are independent from \( \alpha \)-closed, regular semi-closed, ga-closed, \( rga \)-closed, w-closed, rw-closed, sg-closed, pre-closed, g*s-closed, R*-closed, tgr-closed.

Example 4.1 Let \( X = \{a, b, c, d\} \), and \( \tau = \{\emptyset, \{c\}, \{d\}, \{c,d\}, \{a, c, d\}, X\} \). Then

1. \{b,c\} is (gg)*-closed but not \( \alpha \)-closed and \{b\} is \( \alpha \)-closed but not (gg)*-closed.
2. \{b,c,d\} is (gg)*-closed but not ga-closed and \{a\} is ga-closed but not (gg)*-closed.
3. \{a,b\} is (gg)*-closed but not regular semi-closed and \{c\} is regular semi-closed but not (gg)*-closed.
(4) \( \{b, c\} \) is \((gg)^*\) – closed but not \(rg\alpha\)-closed and \( \{a, c, d\} \) is \(rg\alpha\)-closed but not \((gg)^*\) – closed.

(5) \( \{b, d\} \) is \((gg)^*\) – closed but not \(w\)-closed and \( \{b\} \) is \(w\)-closed but not \((gg)^*\) – closed.

(6) \( \{b, c\} \) is \((gg)^*\) – closed but not \(rw\)-closed and \( \{c, d\} \) is \(rw\)-closed but not \((gg)^*\) – closed.

(7) \( \{b, c, d\} \) is \((gg)^*\) – closed but not \(sg\)-closed and \( \{a, d\} \) is \(sg\)-closed but not \((gg)^*\) – closed.

(8) \( \{b, c, d\} \) is \((gg)^*\) – closed but not \(g^*s\)-closed and \( \{d\} \) is \(g^*s\)-closed but not \((gg)^*\) – closed.

(9) \( \{a, b\} \) is \((gg)^*\) – closed but not \(pre\)-closed and \( \{c\} \) is \(pre\)-closed but not \((gg)^*\) – closed.

(10) \( \{b, d\} \) is \((gg)^*\) – closed but not \(R^*\)-closed and \( \{a, c, d\} \) is \(R^*\)-closed but not \((gg)^*\) – closed.

(11) \( \{b, c\} \) is \((gg)^*\) – closed but not \(tgr\)-closed and \( \{c, d\} \) is \(tgr\)-closed but not \((gg)^*\) – closed.

**Remark 4.2**

From the above discussions and known results the relationship between \((gg)^*\)-closed sets and other existing generalizations of closed sets are implemented in Figure: 1

![Figure 1](https://example.com/figure1.png)

In the above figure \( A \rightarrow B \) means the set \( A \) implies the set \( B \) but not conversely and \( A \leftarrow B \) means the set \( A \) and \( B \) are independent of each other.
V. CHARACTERISTICS OF (gg)* - CLOSED SETS

Theorem 5.1 The Union of any two (gg)*-closed sets of X is (gg)*-closed.

Proof: Let A and B be the (gg)*-closed sets in X. Let U be a gg-open set in X such that AU B ⊆ U. Then A ⊆ U and B ⊆ U. Since A and B are (gg)*-closed sets in X, rcl(A) ⊆ U and rcl(B) ⊆ U. We have by [19], rcl(AU B) = rcl(A) ∪ rcl(B) ⊆ U. This implies rcl(AU B) ⊆ U. Hence AU B is (gg)*-closed.

Remark 5.2 Intersection of two (gg)*-closed sets need not be (gg)*-closed as shown in the following example.

Example 5.3 Let X = {a, b, c, d}, and τ = {∅, {c}, {d}, {c, d}, {a, c, d}, X}. Let A = {a, b} and B = {b, c} be two (gg)*-closed sets in X. Then A∩B = {b} is not a (gg)*-closed set.

Theorem 5.4 A subset A of X is (gg)*-closed set in X if and only if rcl(A) – A contains no non-empty gg-closed set.

Proof: Let F be a non-empty gg-closed set in X such that F ⊆ rcl(A) – A. That is F ⊆ rcl(A) ∩ [(A)c]. Therefore F ⊆ rcl(A) and F ⊆ Ac and so A ⊆ Fc. Now since A is (gg)*-closed, and Fc is gg-open, rcl(A) ⊆ Fc. This implies F ⊆ [rcl(A)]c. Also we have F ⊆ rcl(A). Therefore F ⊆ rcl(A) ∩ [rcl(A)]c = ∅. This is a contradiction. Therefore rcl(A) – A contains no non-empty gg-closed set.

Conversely, suppose that rcl(A) – A contains no non-empty gg-closed set. Suppose rcl(A) is not contained in U. Let U be a gg-open set in X such that A ⊆ U. Then rcl(A) ∩ Uc is a non-empty gg-closed set and contained in rcl(A) – A. Which is a contradiction. Hence A is a (gg)*-closed set.

Theorem 5.5 Let A ⊆ B ⊆ rcl(A) and A is (gg)*-closed set in X, then B is also (gg)*-closed.

Proof: Let U be a gg-open set in X such that A ⊆ U. Now if A ⊆ B ⊆ rcl(A), then rcl(A) ⊆ rcl(B) ⊆ rcl(A). Therefore rcl(B) = rcl(A). Since A is (gg)*-closed, rcl(A) ⊆ U. Therefore rcl(B) = rcl(A) ⊆ U. Hence B is (gg)*-closed.

Theorem 5.6 If A is gg-open subset of X and (gg)*-closed set in X. Then A is a regular closed set.

Proof: Since A is gg-open subset of X and (gg)*-closed, rcl(A) ⊆ A. But A ⊆ rcl(A). Therefore A= rcl(A). Hence A is regular closed.

Theorem 5.7 Let A ⊆ B ⊆ X, where B is gg-open and (gg)*-closed in X. If A is (gg)*-closed in B. Then A is (gg)*-closed in X.

Proof: Let U be a gg-open set in X such that A ⊆ U. Since A ⊆ U∩B, U∩B is gg-open in B and A is (gg)*-closed in B, rcl(A) ⊆ U∩B. Now rcl(A)∩B ⊆ U∩B. Since A ⊆ B, rcl(A) ⊆ rcl(B). Since B is gg-open and (gg)*-closed in X, by theorem 5.6, B is regular closed. Therefore rcl(B) = B. This implies rcl(A) ⊆ B. Thus rcl(A) = rcl(A)∩B ⊆ U∩B ⊆ U. Hence A is (gg)*-closed in X.

Theorem 5.8 For every point x of the space X the set X - {x} is either (gg)*-closed (or) gg-open.

Proof: Suppose that X - {x} is not gg-open. Then X is the only gg-open set containing X – {x}. That is X – {x} ⊆ X. This implies rcl(X – {x}) ⊆ rcl(X) ⊆ X. Therefore X – {x} is a (gg)*-closed set in X.

Theorem 5.9
1. If A is β-open and βg*-closed set in X. Then A is (gg)*-closed.
2. If A is R*-open and R* - closed set in X. Then A is (gg)*-closed.
(3) If A is regular open and $r \wedge g$-closed set in X. Then A is $(gg)^*\text{-closed.}$

**Proof:**

(1) Let A be a $\beta$-open and $\beta g^*$-closed set in X. Let U be any gg-open set in X such that $A \subseteq U$. By Definition 2.1(27), $gcl(A) \subseteq A$. Thus we get $rcl(A) \subseteq A$. Therefore $rcl(A) \subseteq U$. Thus we get A is $(gg)^*\text{-closed.}$

(2) Let A be a $R^\ast$-open and $R^\ast$-closed set in X. Let U be any gg-open set in X such that $A \subseteq U$. By Definition 2.1(26), $gcl(A) \subseteq A$. Hence $rcl(A) \subseteq U$. Thus we get A is $(gg)^*\text{-closed.}$

(3) Let A be a regular open and $r \wedge g$-closed set in X. Let U be any gg-open set in X such that $A \subseteq U$. By Definition 2.1(28), $gcl(A) \subseteq A$. Hence $rcl(A) \subseteq U$. Thus we get A is $(gg)^*\text{-closed.}$

**Theorem 5.10**

Let A be a regular semi-open set.

(a) If A is gg-closed then A is $(gg)^*\text{-closed.}$

(b) If A is $R^\ast$-closed then A is $(gg)^*\text{-closed.}$

**Proof:**

(a) Let A be a regular semi-open set and gg-closed set in X. Let U be any gg-open set in X such that $A \subseteq U$. By Definition 2.1(4), $gcl(A) \subseteq A$. Hence $rcl(A) \subseteq U$. Thus we get A is $(gg)^*\text{-closed.}$

(b) Let A be a regular semi-open set and $R^\ast$-closed set in X. Let U be any gg-open set in X such that $A \subseteq U$. By Definition 2.1(25), $gcl(A) \subseteq A$. Hence $rcl(A) \subseteq U$. Thus we get A is $(gg)^*\text{-closed.}$

**VI. CONCLUSION**

The class of $(gg)^*\text{-closed sets} in topological spaces is defined using regular closure and gg-open sets. We have studied the relation of this set with some other closed sets and some of the properties are investigated.

**REFERENCES**


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