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Research Paper

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Some Common Fixed Point Theorems in Complex Valued Intuitionistic Fuzzy Metric Spaces

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Abstract—In this paper, we aim to prove some common fixed point theorems for contraction condition and pairs of occasionally weakly compatible mappings satisfying some conditions in complex valued intuitionistic fuzzy metric spaces.

Keywords—*Common fixed point, Occasionally weakly compatible mappings, Complex valued, Continuous t-norm, Intuitionistic fuzzy set.*

I. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh in 1965 [12]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Since then it has become a vigorous area of research in engineering, medical science, social science, graph theory, metric space theory, complex analysis etc. In 1975, Kramosil and Michalek [6] have introduced the concepts of fuzzy metric spaces in different ways. In 1994, George and Veermani [4] modified the notion of fuzzy metric spaces with the help of continuous t-norms.

The concept of fuzzy complex numbers and fuzzy complex analysis were first introduced by Buckley [3] in 1987. Motivated by the work of Buckley some author's continued re-search in fuzzy complex numbers. In 2002, Ramot et al.[8] extended fuzzy sets to complex fuzzy sets as a generalization. According to Ramot et al., the complex fuzzy set is characterized by a membership function, whose range is not limited to [0, 1] but extended to the unit circle in the complex plane. Later 2016, Singh et al.[10] defined the notion of complex valued fuzzy metric spaces with the help of complex valued continuous t -norm and also defined the notion of convergent sequence, Cauchy sequence in complex valued fuzzy metric spaces. In 1983, Atanassov [1] made an excitement with the introduction of intuitionistic fuzzy sets by adding the idea of non-membership grade to fuzzy set theory.

This paper presents some common fixed point theorems for pairs of occasionally weakly compatible mappings satisfying some conditions in the complex valued intuitionistic fuzzy metric spaces. We also provide some examples which support the main results here.

II. PRELIMINARIES

Definition 2.1: A binary operation $* : r_s e^{i\theta}$ x $r_s e^{i\theta} \to r_s e^{i\theta}$,where $r_s \in [0,1]$ and a fix $\theta \in \left[0,\frac{\pi}{2}\right]$, is called complex valued continuous t-norm if it satisfies the followings:
(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * e^{i\theta} = a,\ \forall\, a \in r_s e^{i\theta}$,
(4) a $*$ b $\precsim c * d$ whenever $a \precsim c$ and $b \precsim d, \forall$ a,b,c,d $\in r_s e^{i\theta}$.

Definition 2.2: A binary operation $\diamond : r_s e^{i\theta}$ x $r_s e^{i\theta} \to r_s e^{i\theta}$,where $r_s \in [0,1]$ and a fix $\theta \in \left[0,\frac{\pi}{2}\right]$, is called complex valued continuous t-conorm if it satisfies the followings:
(1) $\diamond$ is associative and commutative,
(2) $\diamond$ is continuous,
(3) $a \diamond 0 = a,\ \forall\, a \in r_s e^{i\theta}$,
(4) a $\diamond$ b $\precsim c \diamond d$ whenever $a \precsim c$ and $b \precsim d, \forall$ a,b,c,d $\in r_s e^{i\theta}$.

Definition 2.3: The following are examples for complex valued continuous t-norm:

(i) $a * b = min\{a, b\}$, $\forall\, a, b \in r_s e^{i\theta}$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$

(ii) $a * b = max\left(a + b - e^{i\theta}, 0\right)$, for all $a, b \in r_s e^{i\theta}$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$.

Definition 2.4: The following are examples for complex valued continuous t-conorm:

(i) $a \diamond b = max\{a, b\}$, $\forall\, a, b \in r_s e^{i\theta}$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$

(ii) $a \diamond b = min(a + b, 1)$, for all $a, b \in r_s e^{i\theta}$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$.

Definition 2.5: The triplet $(X, M, N, *, \diamond)$ is said to be Complex Valued Intuitionistic Fuzzy Metric Space [*CVIFMS*] if X is an arbitrary non empty set, $*$ is a complex valued continuous t-norm, $\diamond$ is a complex valued continuous t-conorm and $M, N: X \times X \times (0, \infty) \to r_s e^{i\theta}$ are complex valued fuzzy sets, where $r_s \in [0,1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$, satisfying the following conditions: for all x, y, z $\in X$; t, s $\in (0, \infty)$; $r_s \in [0,1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$.

(cf1) $M(x, y, t) + M(x, y, t) \precsim e^{i\theta}$,
(cf2) $M(x, y, t) \succ 0$,
(cf3) $M(x, y, t) = e^{i\theta},\ for\ all\ t \in (0, \infty)$ if and only if x=y,
(cf4) $M(x, y, t) = M(y, x, t)$,
(cf5) $M(x, y, t + s) \succsim M(x, z, t) * M(z, y, s)$,
(cf6) $M(x, y, t) : (0, \infty) \to r_s e^{i\theta}$ is continuous,
(cf7) $N(x, y, t) \prec e^{i\theta}$,
(cf8) $N(x, y, t) = 0, for\ all\ t \in (0, \infty)$ if and only if x=y,
(cf9) $N(x, y, t) = N(y, x, t)$,
(cf10) $N(x, y, t + s) \precsim N(x, z, t) \diamond N(z, y, s)$,
(cf11) $N(x, y, t) : (0, \infty) \to r_s e^{i\theta}$ is continuous,

The pair (M,N) is called a *CVIFMS*. The functions $M(x, y, t)$ and $N(x, y, t)$ denotes the degree of nearness and non-nearness between x and y with respect to t. It is noted that if we take $\theta = 0$, then complex valued intuitionistic fuzzy metric simply goes to real valued intuitionistic fuzzy metric.

Note: It is clear that $r_s e^{i\theta} \precsim e^{i\theta}$ and consequently, $M(x, y, t) \precsim e^{i\theta}$ and $N(x, y, t) \succsim 0$, for all $x, y \in X, t \in (0, \infty), r_s \in [0,1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$.

Example 2.7: Let (X, d) be a real valued metric space. Let $a * b = \min\{a, b\}$ and $a \diamond b = max\{a, b\}$, for all $a, b \in r_s e^{i\theta}$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$.
For each $t > 0, x, y \in X$, we define
$M(x, y, t) = e^{i\theta} \frac{kt^n}{kt^n + md(x,y)}$ and
$N(x, y, t) = \frac{md(x,y)e^{i\theta}}{kt^n + md(x,y)}$, where $k, m, n \in \mathbb{N}$.
Then $(X, M, N, *, \diamond)$ is a *CVIFMS*.
By choosing $k = m = n = 1$, we get
$M(x, y, t) = e^{i\theta} \frac{t}{t + d(x,y)}$ and $N(x, y, t) = \frac{d(x,y)e^{i\theta}}{t + d(x,y)}$.
This *CVIFMS* induced by a metric d is referred to a standard *CVIFMS*.

Example 2.8: Let $X = \mathbb{R}$. Let $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0,1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. For each $t > 0$, $x, y \in X$, we define
$M(x, y, t) = e^{i\theta} e^{\frac{-|x-y|}{t}}$ and
$N(x, y, t) = e^{i\theta}\left(e^{\frac{|x-y|}{t}} - 1\right)\left(e^{\frac{-|x-y|}{t}}\right)$.
Then $(X, M, N, *, \diamond)$ is a *CVIFMS*.

Definition 2.9: Let (X, M,N, *,$\diamond$) be a *CVIFMS*. We define an open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in \mathbb{C}$ with $0 \prec r \prec e^{i\theta}, t > 0$ as

$$B(x, r, t) = \left\{\begin{matrix} y \in X : M(x, y, t) \succ e^{i\theta} - r, \\ N(x, y, t) \prec r \end{matrix}\right\},$$

where $\theta \in \left[0, \frac{\pi}{2}\right]$. A point $x \in X$ is said to be interior point of s set $A \subset X$, whenever there exists $r \in \mathbb{C}$ with $0 \prec r \prec e^{i\theta}$ such that $B(x, r, t) \subset A$, where $\theta \in \left[0, \frac{\pi}{2}\right]$. A subset A of X is called open if every element of a is an interior point of A.
If we define $\tau = \{A \subset X : x \epsilon A\}$ iff there exists $t > 0$ and $r \in \mathbb{C}, 0 \prec r \prec e^{i\theta}$, $\theta \in \left[0, \frac{\pi}{2}\right]$ such that $B(x, r, t) \subset A$. Then one can easily check that τ is a topology on X.

Definition 2.10: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* and τ be the topology induced by *CVIFMS*. Let $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to converges to $x \in X$ if and only if for any $t > 0$, $M(x_n, x, t) \to e^{i\theta}$ or $|M(x_n, x, t)| \to 1$, $N(x_n, x, t) \to 0$ or $|N(x_m, x_n, t)| \to 0$ as $n \to \infty$. A sequence $\{x_n\}$ in a *CVIFMS* (X, M, N, *,$\diamond$) is a Cauchy sequence if and only if for any $t > 0$, $M(x_m, x_n, t) \to e^{i\theta}$, $N(x_m, x_n, t) \to 0$ as $m, n \to \infty$ or $|M(x_m, x_n, t)| \to 1$, $|N(x_m, x_n, t)| \to 0$ as $m, n \to \infty$. A *CVIFMS* in which every Cauchy sequence is convergent is called *CVIFMS*.

For example, let $X = \mathbb{R}$ and $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$, for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0,1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. For each $t > 0$ and $x, y \in X$, we define $M(x, y, t) = \frac{te^{i\theta}}{t + |x-y|}$ and $N(x, y, t) = \frac{|x-y|e^{i\theta}}{t + |x-y|}$.
Then $(X, M, N, *, \diamond)$ is *CVIFMS*.

Definition 2.11: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* and $S, T : X \to X$ be two mappings. A point $x \in X$ is said to be a coincidence point of S and T if and only if $Sx = Tx$. We shall call $w = Sx = Tx$ a point of coincidence of S and T. Moreover, if $Sx = Tx = x$, then the point $x \in X$ is called common fixed point of S and T.

Definition2.12: Let $(X, M, N, *, \diamond)$ be *CVIFMS* and $S, T : X \to X$ be two self mappings. The self maps S and T on X are said to be commuting if $STx = TSx$, for all $x \in X$. The self maps S and T are said to be compatible if

$\lim_{n\to\infty} M(STx_n, TSx_n, t) = e^{i\theta}$ *or*

$\lim_{n\to\infty} |M(STx_n, TSx_n, t)| = 1$ *and*

$\lim_{n\to\infty} N(STx_n, TSx_n, t) = 0$ *or*

$\lim_{n\to\infty} |N(STx_n, TSx_n, t)| = 0,\ t > 0,$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$, for some $x \in X$.

Definition 2.13: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* and $S, T : X \to X$ be two mappings. The self maps S and T on X are said to be weakly compatible if $STx = TSx$whenever $Sx = Tx$, that is, they commute at their coincidence points.

Definition 2.14: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* and $S, T : X \to X$ be two mappings. The self-maps S and T on X are said to be occasionally weakly compatible if and only if there is a coincidence point x in X of S and T at which S and T commute, i.e., $STx = TSx$.

Example 2.15: Let $\mathbb{R}$ be the set of real numbers with standard *CVIFMS*. Define $S, T : \mathbb{R} \to \mathbb{R}$ by $Sx = x^2 + x$ and $Tx = 2x$, for all $x \in \mathbb{R}$. Then $Sx = Tx$, for $x = 0{,}1$ but $ST0 = TS0$ and $ST1 \neq TS1$. Therefore S and T are occasionally weakly compatible self maps but not weakly compatible.

Lemma 2.16: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* such that $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$, $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$. If $M(x, y, kt) \gtrsim M(x, y, t)$ and $N(x, y, kt) \lesssim N(x, y, t)$, for some $0 < k < 1$, for all $x, y \in X, t \in (0, \infty)$, then $x = y$.

Proof: Suppose there exists $k \in (0,1)$, such that $M(x, y, kt) \gtrsim M(x, y, t)$ and $N(x, y, kt) \lesssim N(x, y, t)$ for all $x, y \in X$, $t \in (0, \infty)$. So, that $M(x, y, t) \gtrsim M\left(x, y, \frac{t}{k}\right)$ and $N(x, y, t) \lesssim N\left(x, y, \frac{t}{k}\right)$. Repeated application gives $M(x, y, t) \gtrsim M\left(x, y, \frac{t}{k^n}\right)$ and $N(x, y, t) \lesssim N\left(x, y, \frac{t}{k^n}\right)$ for some positive integer n.
On making $n \to \infty$, reduces to $M(x, y, t) \gtrsim e^{i\theta}$ and $N(x, y, t) \lesssim 0$. This implies $M(x, y, t) = e^{i\theta}$ and $N(x, y, t) = 0$. Thus, we have $x = y$.

Lemma 2. 17: Let $\{x_n\}$ be a sequence in a *CVIFMS* $(X, M, N, *, \diamond)$ with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$ and $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$. If there exists $k \in (0,1)$ such that $M(x_{n+1}, x_{n+2}, kt) \gtrsim M(x_n, x_{n+1}, t)$ and $N(x_{n+1}, x_{n+2}, kt) \lesssim N(x_n, x_{n+1}, t)$ for all $t > 0$ and $n = 0, 1, 2 ....$ Then $\{x_n\}$ is a Cauchy sequence in X.

Proof: For $n = 0$, we have $M(x_1, x_2, t) \gtrsim M\left(x_0, x_1, \frac{t}{k}\right)$ and $N(x_1, x_2, t) \lesssim N\left(x_0, x_1, \frac{t}{k}\right)$, for all $t > 0$ and $k \in (0,1)$. By induction,

$M(x_{n+1}, x_{n+2}, t) \gtrsim M\left(x_0, x_1, \frac{t}{k^{n+1}}\right)$ and

$N(x_{n+1}, x_{n+2}, t) \lesssim N\left(x_0, x_1, \frac{t}{k^{n+1}}\right)$, for all n. Thus for any positive integer p and using (cf5) and (cf10), we have

$$
\begin{aligned}
M(x_n, x_{n+p}, t) &\gtrsim M\left(x_n, x_{n+1}, \tfrac{t}{p}\right) * \ldots *(p \text{ times}) \\
&\quad * \ldots * M\left(x_{n+p-1}, x_{n+p}, \tfrac{t}{p}\right) \\
&\gtrsim M\left(x_0, x_1, \tfrac{t}{pk^n}\right) * \ldots *(p \text{ times}) \\
&\quad * \ldots * M\left(x_0, x_1, \tfrac{t}{pk^{n+p-1}}\right) \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
N(x_n, x_{n+p}, t) &\lesssim N\left(x_n, x_{n+1}, \tfrac{t}{p}\right) \diamond \ldots \diamond (p \text{ times}) \\
&\quad \diamond \ldots \diamond N\left(x_{n+p-1}, x_{n+p}, \tfrac{t}{p}\right) \\
&\lesssim N\left(x_0, x_1, \tfrac{t}{pk^n}\right) \diamond \ldots \diamond (p \text{ times}) \\
&\quad \diamond \ldots \diamond N\left(x_0, x_1, \tfrac{t}{pk^{n+p-1}}\right)
\end{aligned}
$$

which on letting $n \to \infty$, reduces to

$\lim_{n\to\infty} M(x_n, x_{n+p}, t) \gtrsim e^{i\theta} * e^{i\theta} * \ldots e^{i\theta}$ and

$\lim_{n\to\infty} N(x_n, x_{n+p}, t) \lesssim 0 \diamond 0 \diamond \ldots 0$.

Since $k < 1$, $\lim_{n\to\infty} M(x, y, t) = e^{i\theta}$and $\lim_{n\to\infty} N(x, y, t) = 0$, which implies that $\lim_{n\to\infty} M(x_n, x_{n+p}, t) \gtrsim e^{i\theta}$ and $\lim_{n\to\infty} N(x_n, x_{n+p}, t) \lesssim 0$.
This necessitates that $\{x_n\}$ is Cauchy sequence in X.

Lemma 2.18: Let $(X, M, N, *, \diamond)$ be a *CVIFMS*. Let $\{x_n\}$ be a sequence in X converging to $x \in X$. Then any subsequence of $\{x_n\}$ converges to the same point $x \in X$.

III. MAIN RESULTS

Theorem 3.1: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$ and $\lim_{t\to\infty} N(x, y, t) = 0$, for all $x, y \in X$, $t > 0$ and let S and T be self mappings on X. If there exists $k \in (0,1)$ such that $M(Sx, Ty, kt) \gtrsim M(x, y, t)$ and

$$N(Sx, Ty, kt) \lesssim N(x, y, t) \qquad (3.1.1)$$

for all $x, y \in X$ and for all $t > 0$, then S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point and we define the sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$; $n = 0, 1, 2, \ldots$
Now, for $k \in (0,1)$ and for all $t > 0$, we have by the condition (3.1.1)

$$
\begin{aligned}
M(x_{2n+1}, x_{2n+2}, kt) &= M(Sx_{2n}, Tx_{2n+1}, kt) \\
&\gtrsim M(x_{2n}, x_{2n+1}, t), \\
M(x_{2n}, x_{2n+1}, kt) &= M(Sx_{2n-1}, Tx_{2n}, kt)
\end{aligned}
$$

$$\begin{aligned} &\gtrsim M(x_{2n-1}, x_{2n}, t) \quad \text{and} \\ N(x_{2n+1}, x_{2n+2}, kt) &= N(Sx_{2n}, Tx_{2n+1}, kt) \\ &\lesssim N(x_{2n}, x_{2n+1}, t), \\ N(x_{2n}, x_{2n+1}, kt) &= N(Sx_{2n-1}, Tx_{2n}, kt) \\ &\lesssim N(x_{2n-1}, x_{2n}, t). \end{aligned}$$

In general, we have
$M(x_{n+1}, x_{n+2}, kt) \gtrsim M(x_n, x_{n+1}, t)$ and
$N(x_{n+1}, x_{n+2}, kt) \lesssim N(x_n, x_{n+1}, t)$ for all $t > 0$ and $k \in (0,1)$; $n = 0, 1, 2, \ldots$. By Lemma 2.17, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, then there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Obviously $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$ in X, by Lemma 2.18, they are also converge to the same point $u \in X$, $i.e.$, $x_{2n} \to u$ and $x_{2n+1} \to u$ as $n \to \infty$. Now, for this, using condition (3.1.1), we have

$$\begin{aligned} M(Su, u, kt) &= M\left(Su, u, \frac{kt}{2} + \frac{kt}{2}\right) \\ &\gtrsim M\left(Su, x_{2n+2}, \frac{kt}{2}\right) * M\left(x_{2n+2}, u, \frac{kt}{2}\right) \\ &= M\left(Su, Tx_{2n+1}, \frac{kt}{2}\right) * M\left(x_{2n+2}, u, \frac{kt}{2}\right) \\ &\gtrsim M\left(u, x_{2n+1}, \frac{t}{2}\right) * M\left(x_{2n+2}, u, \frac{kt}{2}\right), \end{aligned}$$

$$\begin{aligned} N(Su, u, kt) &= N\left(Su, u, \frac{kt}{2} + \frac{kt}{2}\right) \\ &\lesssim N\left(Su, x_{2n+2}, \frac{kt}{2}\right) \diamond N\left(x_{2n+2}, u, \frac{kt}{2}\right) \\ &= N\left(Su, Tx_{2n+1}, \frac{kt}{2}\right) \diamond N\left(x_{2n+2}, u, \frac{kt}{2}\right) \\ &\lesssim N\left(u, x_{2n+1}, \frac{t}{2}\right) \diamond N\left(x_{2n+2}, u, \frac{kt}{2}\right). \end{aligned}$$

Taking limit as $n \to \infty$, we get
$M(Su, u, kt) \gtrsim e^{i\theta} * e^{i\theta} = e^{i\theta}$ and
$N(Su, u, kt) \lesssim 0 \diamond 0 = 0$.
So, $Su = u$. Again,

$$\begin{aligned} M(u, Tu, kt) &= M\left(u, Tu, \frac{kt}{2} + \frac{kt}{2}\right) \\ &\gtrsim M\left(u, x_{2n+1}, \frac{kt}{2}\right) * M\left(x_{2n+1}, Tu, \frac{kt}{2}\right) \\ &= M\left(u, x_{2n+1}, \frac{kt}{2}\right) * M\left(Sx_{2n}, Tu, \frac{kt}{2}\right) \\ &\gtrsim M\left(u, x_{2n+1}, \frac{t}{2}\right) * M\left(x_{2n}, u, \frac{t}{2}\right), \end{aligned}$$

$$\begin{aligned} N(u, Tu, kt) &= N\left(u, T\,u, \frac{kt}{2} + \frac{kt}{2}\right) \\ &\lesssim N\left(u, x_{2n+1}, \frac{kt}{2}\right) \diamond N\left(x_{2n+1}, Tu, \frac{kt}{2}\right) \\ &= N\left(u, x_{2n+1}, \frac{kt}{2}\right) \diamond N\left(Sx_{2n}, Tu, \frac{kt}{2}\right) \\ &\lesssim N\left(u, x_{2n+1}, \frac{t}{2}\right) \diamond N\left(x_{2n}, u, \frac{t}{2}\right). \end{aligned}$$

Taking limit as $n \to \infty$, we get $M(u, Tu, kt) \gtrsim e^{i\theta} * e^{i\theta} = e^{i\theta}$ and $N(u, Tu, kt) = 0 \diamond 0 = 0$. Therefore $Tu = u$. Thus $Su = Tu = u$ and therefore u is a common fixed point of S and T.
For uniqueness, let z be another fixed point of S and T. Now, using condition (3.1.1),
$M(u, z, kt) = M(Su, Tz, kt) \gtrsim M(u, z, t)$ and
$N(u, z, kt) = N(Su, Tz, kt) \lesssim N(u, z, t)$.
By Lemma 2.16, $u = z$. This completes the theorem.

If we consider $S = T$ in Theorem 3.1, we get the following corollary.

Corollary 3.2: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$ and $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$, $t > 0$ and let S be self mapping on X. If there exists $k \in (0,1)$ such that $M(Sx, Sy, kt) \gtrsim M(x, y, t)$ and $N(Sx, Sy, kt) \lesssim N(x, y, t)$ for all $x, y \in X$ and for all $t > 0$, then S have a unique common fixed point in X.

A supporting example to Corollary 3.2 is given below.

Example 3.3: Let $X = \mathbb{R}$ and $a * b = min\{a, b\}$, $a \diamond b = max\{a, b\}$ for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0, 1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. For each $t > 0$ and $x, y \in X$, we define $M(x, y, t) = \frac{te^{i\theta}}{t + |x-y|}$, $N(x, y, t) = \frac{|x-y|e^{i\theta}}{t + |x-y|}$. Then certainly $(X, M, N, *, \diamond)$ is *CVIFMS* with $\lim\limits_{t\to\infty} M(x, y, t) = e^{i\theta}$ and $\lim\limits_{t\to\infty} N(x, y, t) = 0$.
We define the self-map S on X by $Sx = \frac{x+1}{2}$, for all $x \in X$.
Now, for any $t > 0$ and for $k = \frac{1}{2}$,

$$\begin{aligned} M\left(Sx, Sy, \frac{t}{2}\right) &= \frac{\frac{t}{2} e^{i\theta}}{\frac{t}{2} + |Sx - Sy|} = \frac{te^{i\theta}}{t + 2\left|\frac{x+1}{2} - \frac{y+1}{2}\right|} \\ &= \frac{te^{i\theta}}{t + |x-y|} = M(x, y, t), \end{aligned}$$

$$\begin{aligned} N\left(Sx, Sy, \frac{t}{2}\right) &= \frac{|Sx - Sy| e^{i\theta}}{\frac{t}{2} + |Sx - Sy|} = \frac{\left|\frac{x+1}{2} - \frac{y+1}{2}\right| e^{i\theta}}{\frac{t}{2} + \left|\frac{x+1}{2} - \frac{y+1}{2}\right|} \\ &= \frac{|x-y| e^{i\theta}}{t + |x-y|} = N(x, y, t). \end{aligned}$$

Thus all the conditions of Corollary 3.2 are satisfied and 1 is the unique fixed point of S.

Theorem 3.4: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$, $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$ and let A, B, S and T be self-mappings on X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that

$$M(Ax, By, kt) \gtrsim min \begin{Bmatrix} M(Sx, Ty, t), M(Sx, Ax, t), \\ M(By, Ty, t), \\ M(Ax, Ty, t), M(By, Sx, t) \end{Bmatrix}, \tag{3.4.1}$$

$$N(Ax, By, kt) \lesssim max \begin{Bmatrix} N(Sx, Ty, t), N(Sx, Ax, t), \\ N(By, Ty, t), \\ N(Ax, Ty, t), N(By, Sx, t) \end{Bmatrix} \tag{3.4.2}$$

for all $x, y \in X$ and for all $t > 0$, then A, B, S and T have a unique common fixed point in X.

Proof: Since the pairs $\{A, S\}$ and $\{B, T\}$ are occasionally weakly compatible, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. Now, by the given condition (3.4.1) and (3.4.2) we get

$$M(Ax, By, kt) \gtrsim \min \begin{Bmatrix} M(Sx, Ty, t), M(Sx, Ax, t), \\ M(By, Ty, t), \\ M(Ax, Ty, t), M(By, Sx, t) \end{Bmatrix}$$

$$= \min \begin{Bmatrix} M(Ax,By,t), M(Ax,Ax,t), \\ M(By,By,t), \\ M(Ax,By,t), M(By,Ax,t) \end{Bmatrix}$$
$$= \min \begin{Bmatrix} M(Ax,By,t), e^{i\theta}, e^{i\theta}, \\ M(Ax,By,t), M(By,Ax,t) \end{Bmatrix}$$
$$= M(Ax,By,t),$$

$$N(Ax,By,kt) \precsim max \begin{Bmatrix} N(Sx,Ty,t), N(Sx,Ax,t), \\ N(By,Ty,t), \\ N(Ax,Ty,t), N(By,Sx,t) \end{Bmatrix}$$
$$= \max \begin{Bmatrix} N(Ax,By,t), N(Ax,Ax,t), \\ N(By,By,t), \\ N(Ax,By,t), N(By,Ax,t) \end{Bmatrix}$$
$$= \max \begin{Bmatrix} N(Ax,By,t), 0,0, \\ N(Ax,By,t), N(By,Ax,t) \end{Bmatrix}$$
$$= N(Ax,By,t).$$

In view of Lemma 2.16, we have $Ax = By$ and therefore

$$Ax = Sx = By = Ty. \quad (3.4.3)$$

Suppose that the pair $\{A,S\}$ have an another coincidence point $z \in X$. i.e., $Az = Sz$.

Now,

$$M(Az,By,kt) \succsim min \begin{Bmatrix} M(Sz,Ty,t), M(Sz,Az,t), \\ M(By,Ty,t), \\ M(Az,Ty,t), M(By,Sz,t) \end{Bmatrix}$$
$$= \min \begin{Bmatrix} M(Az,By,t), M(Az,Az,t), \\ M(By,By,t), \\ M(Az,By,t), M(By,Az,t) \end{Bmatrix}$$
$$= \min \begin{Bmatrix} M(Az,By,t), e^{i\theta}, e^{i\theta}, \\ M(Az,By,t), M(By,Az,t) \end{Bmatrix}$$
$$= M(Az,By,t),$$

$$N(Az,By,kt) \precsim max \begin{Bmatrix} N(Sz,Ty,t), N(Sz,Az,t), \\ N(By,Ty,t), \\ N(Az,Ty,t), N(By,Sz,t) \end{Bmatrix}$$
$$= \max \begin{Bmatrix} N(Az,By,t), N(Az,Az,t), \\ N(By,By,t), \\ N(Az,By,t), N(By,Az,t) \end{Bmatrix}$$
$$= \max \begin{Bmatrix} N(Az,By,t), 0,0, \\ N(Az,By,t), N(By,Az,t) \end{Bmatrix}$$
$$= N(Az,By,t).$$

Again, in view of Lemma 2.16, we have $Az = By$. Therefore, $Az = Sz = By = Ty$. (3.4.4)

From (3.4.3) and (3.4.4), $Ax = Az$ and therefore the pair $\{A,S\}$ have a unique point of coincidence $w = Ax = Sx$. Thus by Lemma , w is the unique common fixed point of $\{A,S\}$. Similarly, we can show that the pair $\{B,T\}$ have also a unique common fixed point. Suppose this is $u \in X$.

Now,

$$M(w,u,kt) = M(Aw,Bu,kt)$$
$$\succsim min \begin{Bmatrix} M(Sw,Tu,t), M(Sw,Aw,t), \\ M(Bu,Tu,t), \\ M(Aw,Tu,t), M(Bu,Sw,t) \end{Bmatrix}$$
$$= min \begin{Bmatrix} M(w,u,t), M(w,w,t), \\ M(u,u,t), \\ M(w,u,t), M(u,w,t) \end{Bmatrix}$$
$$= min \begin{Bmatrix} M(w,u,t), e^{i\theta}, e^{i\theta}, \\ M(w,u,t), M(u,w,t) \end{Bmatrix}$$
$$= M(w,u,t) \quad \text{and}$$

$$N(w,u,kt) = N(Aw,Bu,kt)$$
$$\precsim max \begin{Bmatrix} N(Sw,Tu,t), N(Sw,Aw,t), \\ N(Bu,Tu,t), \\ N(Aw,Tu,t), N(Bu,Sw,t) \end{Bmatrix}$$
$$= max \begin{Bmatrix} N(w,u,t), N(w,w,t), \\ N(u,u,t), \\ N(w,u,t), N(u,w,t) \end{Bmatrix}$$
$$= max \begin{Bmatrix} N(w,u,t), 0,0, \\ N(w,u,t), N(u,w,t) \end{Bmatrix}$$
$$= N(w,u,t).$$

Therefore using Lemma 2.16, we have $w = u$ and consequently, w is common fixed point of A, B, S and T. Now,

$$M(w,v,kt) = M(Aw,Bv,kt)$$
$$\succsim min \begin{Bmatrix} M(Sw,Tv,t), M(Sw,Aw,t), \\ M(Bv,Tv,t), \\ M(Aw,Tv,t), M(Bv,Sw,t) \end{Bmatrix}$$
$$= min \begin{Bmatrix} M(w,v,t), M(w,w,t), \\ M(v,v,t), \\ M(w,v,t), M(v,w,t) \end{Bmatrix}$$
$$= min \begin{Bmatrix} M(w,v,t), e^{i\theta}, e^{i\theta}, \\ M(w,v,t), M(v,w,t) \end{Bmatrix}$$
$$= M(w,v,t),$$

$$N(w,v,kt) = N(Aw,Bv,kt)$$
$$\precsim max \begin{Bmatrix} N(Sw,Tv,t), N(Sw,Aw,t), \\ N(Bv,Tv,t), \\ N(Aw,Tv,t), N(Bv,Sw,t) \end{Bmatrix}$$
$$= max \begin{Bmatrix} N(w,v,t), N(w,w,t), \\ N(v,v,t), \\ N(w,v,t), N(v,w,t) \end{Bmatrix}$$
$$= min \begin{Bmatrix} N(w,v,t), 0,0, \\ N(w,v,t), N(v,w,t) \end{Bmatrix}$$
$$= N(w,v,t).$$

By Lemma 2.16, we have $w = v$. Hence A, B, S and T have a unique common fixed point.

In the following an supporting example to Theorem 3.4 is given.

Example 3.5: Let $X = \mathbb{R}$. Consider the metric $d(x,y) = |x| + |y|$, for all $x \neq y$ and $d(x,y) = 0$, for $x = y$ on X. Let $a * b = min\{a,b\}$ and$a \diamond b = max\{a,b\}$ for all $a, b \in r_s e^{i\theta}$, where $r_s \in [0,1]$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. For each $t > 0$ and $x, y \in X$, we define

$M(x,y,t) = \frac{te^{i\theta}}{t+d(x,y)}$, $N(x,y,t) = \frac{d(x,y)e^{i\theta}}{t+d(x,y)}$.

Then certainly $(X, M, N, *, \diamond)$ is *CVIFMS* with $\lim_{t\to\infty} M(x,y,t) = e^{i\theta}$and $\lim_{t\to\infty} N(x,y,t) = 0$.

Now, we define the self maps A, B, S and T on X by $Ax = \frac{x}{3}, Bx = \frac{x}{4}, Sx = x$ and $Tx = \frac{x}{2}$, for all $x \in X$. Let $k = \frac{1}{2}$.

Now, for $x \neq y$,

$$M\left(Ax, By, \frac{t}{2}\right) = \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2}+|Ax|+|By|} = \frac{te^{i\theta}}{t+2\frac{|x|}{3}+2\frac{|y|}{4}} \gtrsim \frac{te^{i\theta}}{t+2\frac{|x|}{2}+2\frac{|y|}{4}}$$
$$= \frac{te^{i\theta}}{t+|x|+\frac{|y|}{2}} = \frac{te^{i\theta}}{t+|Sx|+|Ty|} = M(Sx, Ty, t),$$

$$N\left(Ax, By, \frac{t}{2}\right) = \frac{(|Ax|+|By|)e^{i\theta}}{\frac{t}{2}+|Ax|+|By|} = \frac{\left(\left|\frac{x}{3}\right|+\left|\frac{y}{4}\right|\right)e^{i\theta}}{\frac{t}{2}+\left(\left|\frac{x}{3}\right|+\left|\frac{y}{4}\right|\right)}$$
$$\lesssim \frac{\left(2\left|\frac{x}{3}\right|+2\left|\frac{y}{4}\right|\right)e^{i\theta}}{t+\left(2\left|\frac{x}{3}\right|+2\left|\frac{y}{4}\right|\right)} \lesssim \frac{\left(|x|+\left|\frac{y}{2}\right|\right)e^{i\theta}}{t+\left(|x|+\left|\frac{y}{2}\right|\right)} = \frac{(|Sx|+|Ty|)e^{i\theta}}{t+(|Sx|+|Ty|)}$$
$$= N(Sx, Ty, t).$$

For the case $x = y$.

$$M\left(Ax, By, \frac{t}{2}\right) = \frac{\frac{t}{2}e^{i\theta}}{\frac{t}{2}+0} = e^{i\theta} = M(Sx, Ty, t) \text{ and}$$
$$N\left(Ax, By, \frac{t}{2}\right) = \frac{e^{i\theta}.0}{\frac{t}{2}+0} = 0 = N(Sx, Ty, t).$$

Therefore, for any $x, y \in X$,

$$M\left(Ax, By, \frac{t}{2}\right) \gtrsim M(Sx, Ty, t)$$
$$= \min\left\{\begin{matrix} M(Sx, Ty, t), M(Sx, Ax, t), \\ M(By, Ty, t), \\ M(Ax, Ty, t), M(By, Sx, t) \end{matrix}\right\},$$
$$N\left(Ax, By, \frac{t}{2}\right) \lesssim N(Sx, Ty, t)$$
$$= max\left\{\begin{matrix} N(Sx, Ty, t), N(Sx, Ax, t), \\ N(By, Ty, t), \\ N(Ax, Ty, t), N(By, Sx, t) \end{matrix}\right\}$$

Therefore, the maps A, B, S and T satisfies the condition (3.4.1) of Theorem 3.4 for $k = \frac{1}{2}$. Also, the pairs $\{A, S\}$ and $\{B, T\}$ are obviously occasionally weakly compatible. Thus all the conditions of Theorem 3.4 are satisfied and $x = 0$ is the unique common fixed point of A, B, S and T in X.

Setting $A = B$ and $S = T$ in Theorem 3.4, we get the following corollary.

Corollary 3.6: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$ and $\lim_{t\to\infty} N(x, y, t) = 0$, for all $x, y \in X$ and let A and S be self-mappings on X. Let $\{A, S\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that

$$M(Ax, Ay, kt) \gtrsim min\left\{\begin{matrix} M(Ax, Sy, t), M(Ay, Sx, t), \\ M(Sx, Sy, t), \\ M(Sx, Ax, t), M(Sy, Ay, t) \end{matrix}\right\},$$
$$N(Ax, Ay, kt) \lesssim max\left\{\begin{matrix} N(Ax, Sy, t), N(Ay, Sx, t), \\ N(Sx, Sy, t), \\ N(Sx, Ax, t), N(Sy, Ay, t) \end{matrix}\right\},$$

for all $x, y \in X$ and for all $t > 0$, then A and S have a unique common fixed point in X.

Theorem 3.7: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$ and $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$ and let A, B, S and T be self-mappings on X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that

$$M(Ax, By, kt) \gtrsim g\left(min\left\{\begin{matrix} M(Sx, Ty, t), \\ M(Sx, Ax, t), \\ M(By, Ty, t), \\ M(Ax, Ty, t), \\ M(By, Sx, t) \end{matrix}\right\}\right),$$
$$N(Ax, By, kt) \lesssim g\left(max\left\{\begin{matrix} N(Sx, Ty, t), \\ N(Sx, Ax, t), \\ N(By, Ty, t), \\ N(Ax, Ty, t), \\ N(By, Sx, t) \end{matrix}\right\}\right)$$

for all $x, y \in X$ and for all $t > 0$, where $g : r_s e^{i\theta} \to r_s e^{i\theta}$ with $g(x) \gtrsim x$ for all $x \in r_s e^{i\theta}$, where $r_s \in (0,1)$ and $\theta \in \left[0, \frac{\pi}{2}\right]$. Then A, B, S and T have a unique common fixed point in X.

Proof: The proof follows from Theorem 3.4.

Theorem 3.8: Let $(X, M, N, *, \diamond)$ be a *CVIFMS* with $\lim_{t\to\infty} M(x, y, t) = e^{i\theta}$ and $\lim_{t\to\infty} N(x, y, t) = 0$, for all $x, y \in X$ and let A, B, S and T be self - mappings on X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that

$$M(Ax, By, kt) \gtrsim \left\{\begin{matrix} M(Sx, Ty, t) * M(Ax, Sx, t) * \\ M(By, Ty, t) * M(Ax, Ty, t) \end{matrix}\right\} \quad (3.8.1)$$
$$N(Ax, By, kt) \lesssim \left\{\begin{matrix} N(Sx, Ty, t) \diamond N(Ax, Sx, t) \diamond \\ N(By, Ty, t) \diamond N(Ax, Ty, t) \end{matrix}\right\} \quad (3.8.2)$$

for all $x, y \in X$ and for all $t > 0$. Then A, B, S and T have a unique common fixed point in X.

Proof: The pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. Now, the condition (3.8.1) and (3.8.2) gives

$$M(Ax, By, kt) \gtrsim \left\{\begin{matrix} M(Sx, Ty, t) * M(Ax, Sx, t) * \\ M(By, Ty, t) * M(Ax, Ty, t) \end{matrix}\right\},$$
$$= \left\{\begin{matrix} M(Ax, By, t) * M(Ax, Ax, t) * \\ M(By, By, t) * M(Ax, By, t) \end{matrix}\right\},$$
$$= \left\{\begin{matrix} M(Ax, By, t) * e^{i\theta} * e^{i\theta} * \\ M(Ax, By, t) \end{matrix}\right\},$$
$$= M(Ax, By, t) \quad \text{and}$$

$$N(Ax, By, kt) \lesssim \left\{\begin{matrix} N(Sx, Ty, t) \diamond N(Ax, Sx, t) \diamond \\ N(By, Ty, t) \diamond N(Ax, Ty, t) \end{matrix}\right\},$$
$$= \left\{\begin{matrix} N(Ax, By, t) \diamond N(Ax, Ax, t) \diamond \\ N(By, By, t) \diamond N(Ax, By, t) \end{matrix}\right\},$$
$$= N(Ax, By, t) \diamond 0 \diamond 0 \diamond N(Ax, By, t),$$
$$= N(Ax, By, t).$$

In view of Lemma 2.17, we have $Ax = By$ and therefore

$$Ax = Sx = By = Ty. \quad (3.8.3)$$

Suppose that the pair $\{A,S\}$ have an another coincidence point $z\in X$, i.e., $Az=Sz$.

$$M(Az,By,kt)\gtrsim\begin{Bmatrix}M(Sz,Ty,t)*\ M(Az,Sz,t)*\\ M(By,Ty,t)*M(Az,Ty,t)\end{Bmatrix},$$
$$=\begin{Bmatrix}M(Az,By,t)*\ M(Az,Az,t)*\\ M(By,By,t)*M(Az,By,t)\end{Bmatrix},$$
$$=\begin{Bmatrix}M(Az,By,t)*\ e^{i\theta}*\ e^{i\theta}*\\ M(Az,By,t)\end{Bmatrix},$$
$$=\ M(Az,By,t)\ \text{ and}$$

$$N(Az,By,kt)\lesssim\begin{Bmatrix}N(Sz,Ty,t)\diamond\ N(Az,Sz,t)\diamond\\ N(By,Ty,t)\diamond N(Az,Ty,t)\end{Bmatrix},$$
$$=\begin{Bmatrix}N(Az,By,t)\diamond\ N(Az,Az,t)\diamond\\ N(By,By,t)\diamond N(Az,By,t)\end{Bmatrix},$$
$$=\ N(Az,By,t)\diamond\ 0\diamond\ 0\diamond\ N(Az,By,t),$$
$$=\ N(Az,By,t).$$

By lemma 2.17, $Az=By$ and consequently

$$Az=Sz=By=Ty. \qquad (3.8.4)$$

From (3.8.3) and (3.8.4) $Ax=Az$ and therefore the pair $\{A,S\}$ have a unique point of coincidence $w=Ax=Sx$. w is the unique common fixed point of $\{A,S\}$. Similarly, we can show that there is a unique common fixed point $u\in X$ of $\{B,T\}$.

Now,

$$M(w,u,kt)=M(Aw,Bu,kt)$$
$$\gtrsim\begin{Bmatrix}M(Sw,Tu,t)*M(Aw,Sw,t)*\\ M(Bu,Tu,t)*M(Aw,Tu,t)\end{Bmatrix}$$
$$=\begin{Bmatrix}M(Aw,Bu,t)*\ M(Aw,Aw,t)*\\ M(Bu,Bu,t)*M(Aw,Bu,t)\end{Bmatrix}$$
$$=\begin{Bmatrix}M(Aw,Bu,t)*\ e^{i\theta}*\ e^{i\theta}*\\ M(Aw,Bu,t)\end{Bmatrix}$$
$$=\ M(Aw,Bu,t)=M(w,u,t),$$

$$N(w,u,kt)=N(Aw,Bu,kt)$$
$$\lesssim\begin{Bmatrix}N(Sw,Tu,t)\diamond N(Aw,Sw,t)\diamond\\ N(Bu,Tu,t)\diamond N(Aw,Tu,t)\end{Bmatrix}$$
$$=\begin{Bmatrix}N(Aw,Bu,t)\diamond\ N(Aw,Aw,t)\diamond\\ N(Bu,Bu,t)\diamond N(Aw,Bu,t)\end{Bmatrix}$$
$$=\ N(Aw,Bu,t)\diamond 0\diamond\ 0\diamond\ N(Aw,Bu,t)$$
$$=\ N(Aw,Bu,t)=N(w,u,t).$$

By Lemma 2.17, we have $w=u$ and consequently w is common fixed point of A,B,S and T. For uniqueness, let v is an another common fixed point of A,B,S and T.

Therefore,

$$M(w,v,kt)=M(Aw,Bv,kt),$$
$$\gtrsim\begin{Bmatrix}M(Sw,Tv,t)*\ M(Aw,Sw,t)*\\ M(Bv,Tv,t)*M(Aw,Tv,t)\end{Bmatrix},$$
$$=\begin{Bmatrix}M(w,v,t)*\ M(w,w,t)*\\ M(v,v,t)*M(w,v,t)\end{Bmatrix},$$
$$=\ M(w,v,t)*\ e^{i\theta}*\ e^{i\theta}*M(w,v,t)$$
$$=\ M(w,v,t),$$
$$N(w,v,kt)=N(Aw,Bv,kt)$$
$$\lesssim\begin{Bmatrix}N(Sw,Tv,t)\diamond\ N(Aw,Sw,t)\diamond\\ N(Bv,Tv,t)\diamond N(Aw,Tv,t)\end{Bmatrix}$$
$$=\begin{Bmatrix}N(w,v,t)\diamond\ N(w,w,t)\diamond\\ N(v,v,t)\diamond N(w,v,t)\end{Bmatrix}$$
$$=N(w,v,t)\diamond\ 0\diamond\ 0\diamond N(w,v,t)$$
$$=N(w,v,t)$$

and in view of Lemma 2.17, we have $w=v$. Hence A,B,S and T have a unique common fixed point.

A supporting example to Theorem 3.8 is given below.

Example 3.9: Let $X=\mathbb{R}$.Consider the metric $d(x,y)=|x|+|y|$, for all $x\neq y$ and $d(x,y)=0$, for $x=y$ on X. Let $a*b=min\{a,b\}$, $a\diamond b=max\{a,b\}$ for all $a,b\in r_s e^{i\theta}$, where $r_s\in[0,1]$ and $\theta\in\left[0,\frac{\pi}{2}\right]$. For each $t>0$ and $x,y\in X$, we define

$M(x,y,t)=e^{i\theta}\frac{t}{t+d(x,y)}$ and $N(x,y,t)=e^{i\theta}\frac{d(x,y)}{t+d(x,y)}$.

Then certainly $(X,M,N,*,\diamond)$ is *CVIFMS* with $\lim_{t\to\infty}M(x,y,t)=e^{i\theta}$, $\lim_{n\to\infty}N(x,y,t)=0$.

Now, we define the self maps A,B,S and T on X by $Ax=\frac{x}{5},Bx=\frac{x}{15},Sx=x$ and $Tx=\frac{x}{4}$, for all $x\in X$. Let $k=\frac{1}{3}$. Now, for $x\neq y$,

$$M\left(Ax,By,\frac{t}{3}\right)=\frac{\frac{t}{3}e^{i\theta}}{\frac{t}{3}+|Ax|+|By|}=\frac{te^{i\theta}}{t+2\frac{|x|}{3}+2\frac{|y|}{4}}$$
$$\gtrsim\frac{te^{i\theta}}{t+3\frac{|x|}{5}+3\frac{|y|}{15}}=\frac{te^{i\theta}}{t+|x|+\frac{|y|}{4}}=\frac{te^{i\theta}}{t+|Sx|+|Ty|}$$
$$=M(Sx,Ty,t)$$

$$N\left(Ax,By,\frac{t}{3}\right)=\frac{(|Ax|+|By|)e^{i\theta}}{\frac{t}{3}+(|Ax|+|By|)}=\frac{\left(2\frac{|x|}{3}+2\frac{|y|}{4}\right)e^{i\theta}}{t+2\frac{|x|}{3}+2\frac{|y|}{4}}$$
$$=\frac{\left(3\frac{|x|}{5}+3\frac{|y|}{15}\right)e^{i\theta}}{t+\left(3\frac{|x|}{5}+3\frac{|y|}{15}\right)}\lesssim\frac{\left(|x|+\frac{|y|}{4}\right)e^{i\theta}}{t+|x|+\frac{|y|}{4}}$$
$$=\frac{(|Sx|+|Ty|)e^{i\theta}}{t+|Sx|+|Ty|}=N(Sx,Ty,t).$$

For the case $x=y$.

$$M\left(Ax,By,\frac{t}{3}\right)=\frac{\frac{t}{3}e^{i\theta}}{\frac{t}{3}+0}=e^{i\theta}=M(Sx,Ty,t).$$
$$N\left(Ax,By,\frac{t}{3}\right)=\frac{e^{i\theta}.0}{\frac{t}{3}+0}=0=N(Sx,Ty,t).$$

Therefore, for any $x,y\in X$,

$$M\left(Ax,By,\frac{t}{3}\right)\gtrsim M(Sx,Ty,t)$$
$$=\min\begin{Bmatrix}M(Sx,Ty,t),M(Sx,Ax,t),\\ M(By,Ty,t),\\ M(Ax,Ty,t),M(By,Sx,t)\end{Bmatrix},$$
$$N\left(Ax,By,\frac{t}{3}\right)\lesssim N(Sx,Ty,t)$$
$$=\max\begin{Bmatrix}N(Sx,Ty,t),N(Sx,Ax,t),\\ N(By,Ty,t),\\ N(Ax,Ty,t),N(By,Sx,t)\end{Bmatrix}$$

Therefore, the maps A,B,S and T satisfies the condition (3.8.1) and (3.8.2) of Theorem 3.8 for $k=\frac{1}{3}$. Also the pairs $\{A,S\}$ and $\{B,T\}$ are obviously occasionally weakly compatible. Thus all the conditions of Theorem 3.7 are satisfied and $x=0$ is the unique common fixed point of A,B,S and T in X.

Setting $A=B$ and $S=T$ in Theorem 3.8, we get the following corollary.

Corollary 3.10: Let $(X, M, N, *, \diamond)$ is *CVIFMS* with $\lim_{t\to\infty} M(x,y,t) = e^{i\theta}$ and $\lim_{t\to\infty} N(x,y,t) = 0$ for all $x, y \in X$ and let A, S be self-mappings on X. Let the pairs $\{A, S\}$ be occasionally weakly compatible. If there exists $k \in (0,1)$ such that

$M(Ax, Ay, kt) \gtrsim \begin{Bmatrix} M(Sx,Sy,t) * M(Ax,Sx,t) * \\ M(Ay,Sy,t) * M(Ax,Sy,t) \end{Bmatrix}$ and

$N(Ax, Ay, kt) \lesssim \begin{Bmatrix} N(Sx,Sy,t) \diamond N(Ax,Sx,t) \diamond \\ N(Ay,Sy,t) \diamond N(Ax,Sy,t) \end{Bmatrix}$,

for all $x, y \in X$ and for all $t > 0$, then A and S have a unique common fixed point in X.

Theorem 3.11: Let $(X, M, N, *, \diamond)$ is *CVIFMS* with $\lim_{t\to\infty} M(x,y,t) = e^{i\theta}$ and$\lim_{t\to\infty} N(x,y,t) = 0$ for all $x, y \in X$. Let the pairs $\{A, S\}$ be occasionally weakly compatible self-maps on X. If there exists $k \in (0,1)$ such that

$$M(Sx,Sy,kt) \gtrsim pM(Ax,Ay,t) + q\, min \begin{Bmatrix} M(Ax,Ay,t), \\ M(Sx,Ax,t), \\ M(Sy,Ay,t) \end{Bmatrix} \text{ and}$$

$$N(Sx,Sy,kt) \lesssim pN(Ax,Ay,t) + q\, max \begin{Bmatrix} N(Ax,Ay,t), \\ N(Sx,Ax,t), \\ N(Sy,Ay,t) \end{Bmatrix} \qquad (3.11.1)$$

for all $x, y \in X$ and for all $t > 0$ where $p, q > 0$ are real with $p + q \geq 1$. Then A and S have a unique common fixed point in X.

Proof: The pair are occasionally weakly compatible, then there exists $x \in X$ such that $Ax = Sx$. Suppose that there exists another $y \in X$ for which $Ay = Sy$. From the condition (3.11.1),

$$\begin{aligned} M(Sx,Sy,kt) &\gtrsim pM(Ax,Ay,t) + q\, min \begin{Bmatrix} M(Ax,Ay,t), \\ M(Sx,Ax,t), \\ M(Sy,Ay,t) \end{Bmatrix} \\ &= pM(Sx,Sy,t) + q\, min \begin{Bmatrix} M(Sx,Sy,t), \\ M(Sx,Sx,t), \\ M(Sy,Sy,t) \end{Bmatrix} \\ &= pM(Sx,Sy,t) + q\, min \begin{Bmatrix} M(Sx,Sy,t), \\ e^{i\theta}, e^{i\theta} \end{Bmatrix} \\ &= pM(Sx,Sy,t) + qM(Sx,Sy,t) \\ &= (p+q)M(Sx,Sy,t). \end{aligned}$$

Since $p + q \geq 1, M(Sx,Sy,kt) \gtrsim M(Sx,Sy,t)$,

$$\begin{aligned} N(Sx,Sy,kt) &\lesssim pN(Ax,Ay,t) + q\, max \begin{Bmatrix} N(Ax,Ay,t), \\ N(Sx,Ax,t), \\ N(Sy,Ay,t) \end{Bmatrix} \\ &= pN(Sx,Sy,t) + q\, max \begin{Bmatrix} N(Sx,Sy,t), \\ N(Sx,Sx,t), \\ N(Sy,Sy,t) \end{Bmatrix} \\ &= pN(Sx,Sy,t) + q\, max \begin{Bmatrix} N(Sx,Sy,t), \\ 0,0 \end{Bmatrix} \\ &= pN(Sx,Sy,t) + qN(Sx,Sy,t) \\ &= (p+q)N(Sx,Sy,t). \end{aligned}$$

Since $p + q \geq 1, N(Sx,Sy,kt) \lesssim N(Sx,Sy,t)$.

In view of Lemma 2.17, we have $Sx = Sy$ and consequently $Ax = Ay$. Therefore the pair $\{A, S\}$ have a unique point of coincidence $w = Ax = Sx$. Thus, A and S have a unique common fixed point in X.

The following corollary is the direct consequence of Theorem 3.11.

Corollary 3.12: Let $(X, M, N, *, \diamond)$ is *CVIFMS* with $\lim_{t\to\infty} M(x,y,t) = e^{i\theta}$ and $\lim_{t\to\infty} N(x,y,t) = 0$ for all $x, y \in X$. Let A be self map on X. If there exists $k \in (0,1)$ such that $M(Ax,Ay,kt) \gtrsim \alpha\, M(Ax,Ay,t)$ and $N(Ax,Ay,kt) \lesssim \alpha\, N(Ax,Ay,t)$, for all $x, y \in X$ and for all $t > 0$, where $\alpha \geq 1$ real. Then A has a unique fixed point in X.

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