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An Examination of the Accuracy and Zero Stability of the Explicit Linear Two-Step Method for Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs)

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***Abstract*—**In this paper, an explicit linear two-step method of maximal order containing one free parameter for the solution of IVPs in ODEs is presented. The performance measure of the method in terms of the accuracy and zero stability is examined. The bound of the local truncation error for the explicit linear one-step method has been investigated. Numerical example has been solved successfully via the explicit linear two-step method by varying the free parameter. The results obtained show that the explicit linear two-step method is zero stable and agrees with the exact solution. In the case of *b = -5*, the method is zero unstable. It can also be concluded that one order decrease in the values of the step length leads to third order decrease in the magnitude of the error bound of the method. The methodology can be applied to the solution of higher order ODEs emanated from real life situations with points of catastrophe.

Keywords—Bound, Explicit linear two-step method, Initial value problem, Local truncation error, Exact solution

I. INTRODUCTION

Most of the models emanated from finance, science, engineering, social science, medicine which are formulated by means of the differential equations are so complicated to determine their exact solutions. There had been a lot of developed methods for the numerical solutions of IVPs in ODEs such as [1-13], just to mention a few. The main aim of this paper is to examine the performance of the explicit linear two-step method of maximal order containing one free parameter in terms of accuracy and zero stability. The rest of the paper is organized as follows: Section Two presents the overview of some related research works. Chapter Three is the methodology. Section Four consists of the implementation of the explicit linear two-step method on IVP in ODEs. The bound on the local truncation error for the method is also presented. Section Five presents the results and discussion. The concluding remarks and the future scope were captured by Section Six.

II. RELATED WORK

Some approaches for solving system of ordinary differential equations directly have been suggested and developed by many authors. A zero stable method for direct solution of fourth order ordinary differential equations has been developed by [14]. Kayode [15] considered A zero-stable optimal order method for direct solution of second order differential equations. A new sixth-order algorithm for general second order ordinary differential equations was proposed and developed by [16]. Awoyemi [17] presented a class of continuous linear methods for the solution of general second order initial value problems in ordinary differential equations. Fadugba [18] constructed an explicit linear two-step method of maximal order. He also discussed extensively; the properties of the method such as order of accuracy, error constant, consistency, zero stability, convergence and the local truncation error.

In this paper, by varying the free parameter, the explicit linear two-step method shall be implemented on IVP in ODEs and the results shall be compared with the exact solutions.

III. METHODOLOGY

In this section, an explicit linear two-step method of maximal order containing one free parameter is presented [18]. Here, $k=2, \alpha_2=1, \beta_2=0$, by hypothesis. Let $\alpha_0=b$ be the free parameter. There remain three undetermined coefficients $\alpha_1, \beta_0, \beta_1$. From the C_q constants, one can write that

$$C_q=\sum_{j=0}^{2}\left(\frac{j^q}{q!}\alpha_j-\frac{j^{(q-1)}}{(q-1)!}\beta_j\right) \quad (1)$$

$$C_0 = \sum_{j=0}^{2} \alpha_j = \alpha_1 + b + 1 = 0 \qquad (2)$$

$$C_1 = \sum_{j=0}^{2} \left(j\alpha_j - \beta_j \right) = \alpha_1 + 2 - \beta_0 - \beta_1 = 0 \qquad (3)$$

$$C_2 = \sum_{j=0}^{2} \left(\frac{j^2}{2}\alpha_j - j\beta_j \right) = \left(\frac{\alpha_1 + 4}{2} \right) - \beta_1 = 0 \qquad (4)$$

From (2),

$$\alpha_1 = -(b+1) \qquad (5)$$

Also from (3), one gets

$$\alpha_1 - \beta_0 - \beta_1 = -2 \qquad (6)$$

Using (5) and (6), yields

$$\beta_0 + \beta_1 = 1 - b \qquad (7)$$

Substituting (5) into (4) gives

$$\beta_1 = \left(\frac{3-b}{2} \right) \qquad (8)$$

To get β_0, combining (7) and (8) yields

$$\beta_0 = -\left(\frac{1+b}{2} \right) \qquad (9)$$

Hence,

$$\alpha_0 = b, \alpha_1 = -(b+1), \alpha_2 = 0, \beta_0 = -\left(\frac{1+b}{2} \right), \beta_1 = \left(\frac{3-b}{2} \right), \beta_2 = 0 \qquad (10)$$

From the general linear two-step method, one obtains

$$\sum_{j=0}^{2} \alpha_j y_{n+j} = h \sum_{j=0}^{2} \beta_j f_{n+j} \qquad (11)$$

Substituting (10) into (11) and rearranging terms, yields

$$y_{n+2} - (1+b) y_{n+1} + b y_n = \frac{h}{2}\left((3-b) f_{n+1} - (1+b) f_n \right) \qquad (12)$$

Equation (12) is the required mehod.

Remarks

The following are the properties of the derived method [18]

i) For *b* = 0, the method (12) is of order two. The error constant is obtained as $C_3 = 0.4167$.

ii) For *b* = 0, the method (12) is zero stable.

iii) For *b* = -5, the method (12) is of the order three and the error constant is obtained as $C_4 = 0.1667$.

iv) For *b* = -5, the method (12) is not zero stable.

v) Equation (12) is consistent for *b* = 0 and *b* = -5, since $C_0 = C_1 = 0$.

vi) In the case where $b = 0$, (12) is convergent whereas in the case $b = -5$, (12) diverges.

vii) The following result gives the Local Truncation Error (LTE) for the general linear two-step method under the localizing assumption.

Theorem 2.1: If the real valued function y is of class $C^2[a,b]$, then the local truncation error $L[y(x_n);h]$ associated with the general linear two-step method

$$\sum_{j=0}^{2} \alpha_j y_{n+j} = h \sum_{j=0}^{2} \beta_j f_{n+j} \qquad (13)$$

for the numerical solution of the initial value problem

$$y' = f(x,y),\ y(x_0) = y_0,\ x_0 \le x \le b,\ y \in \Re \qquad (14)$$

satisfies

$$L\left[y(x_n);h\right] = \left[1 - h\beta_2 \frac{\partial f\left(x_{n+2}, \zeta_{n+2}\right)}{\partial y} \right] \left(y\left(x_{n+2}\right) - y_{n+2} \right) \qquad (15)$$

where $\zeta_{n+2} \in \left(y_{n+2}, y(x_{n+2}) \right)$.

i) Equation (15) is the LTE for the implicit linear two-step method, the case where $\beta_2 \ne 0$.

ii) For an explicit case, that is; for $\beta_2 = 0$, the LTE is the difference between the exact solution and the numerical solution. This can be expressed as

$$T_{n+2} = L(y(x_n);h) = y(x_{n+2}) - y_{n+2} \qquad (16)$$

IV. IMPLEMENTATION OF THE METHOD

Consider the initial value problem of the form

$$y' = 4xy^{\frac{1}{2}},\ y(0) = 1 \qquad (17)$$

whose exact solution is obtained as

$$y(x) = (1+x^2)^2 \qquad (18)$$

Varying the free parameter, the results generated via the explicit linear two-step method (ETSM) in the context of the exact solution (ES) were displayed in the Figures 1 and 3. The errors generated via the ETSM for *b = 0* and *b = -5* are displayed in Figures 2 and 4, respectively. The comparative results analysis of ETSM and ES for different values of the free parameter is shown in Figure 5. The relative errors of ETSM for different values of the free parameters are shown in Figure 6. The bound for the local truncation error of the derived two-step explicit method with *b = 0* is presented below.

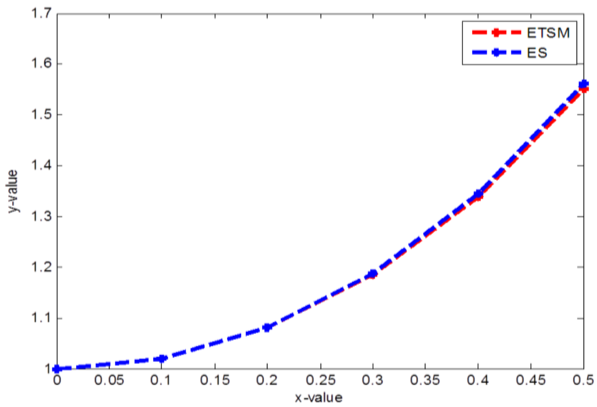

Figure 1: The comparative results analysis of the explicit linear two-step method and the exact solution for *b* = 0.

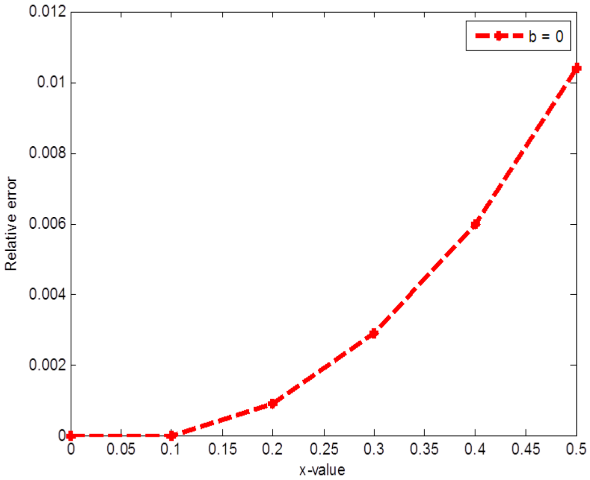

Figure 2: The relative error generated via the explicit linear two-step method for $b = 0$.

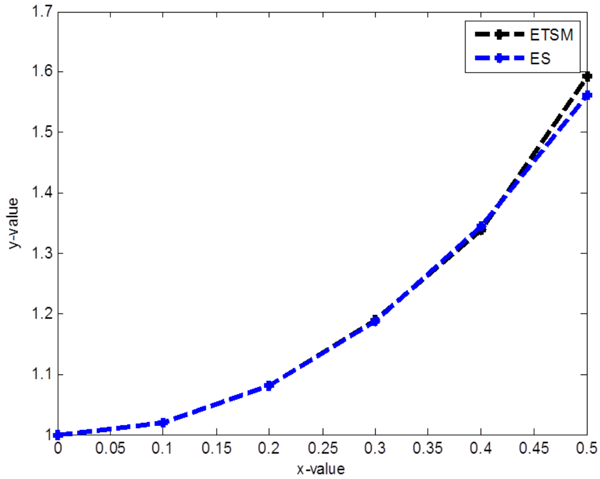

Figure 3: The comparative results analysis of the explicit linear two-step method and the exact solution for $b = -5$.

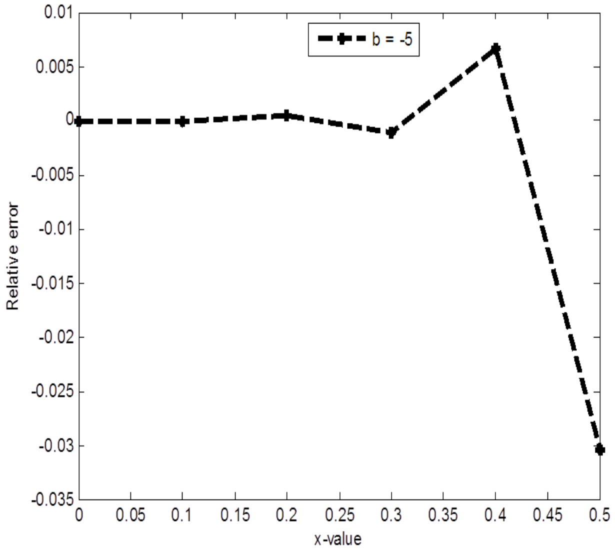

Figure 4: The relative error generated via the explicit linear two-step method for $b = -5$.

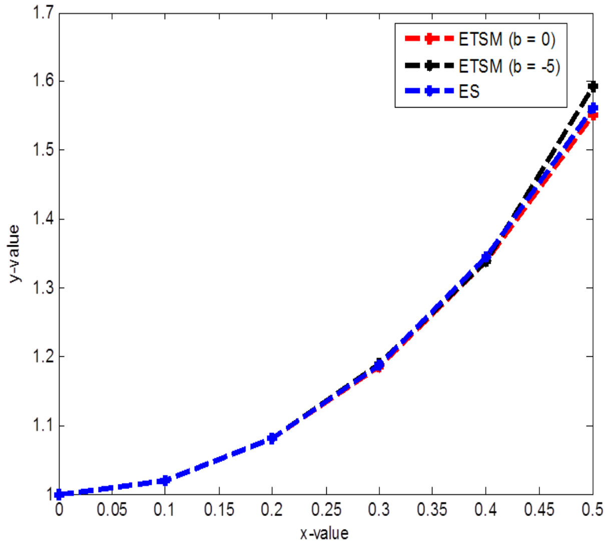

Figure 5: The comparative results analysis of the explicit linear two-step method and the exact solution for different values of b

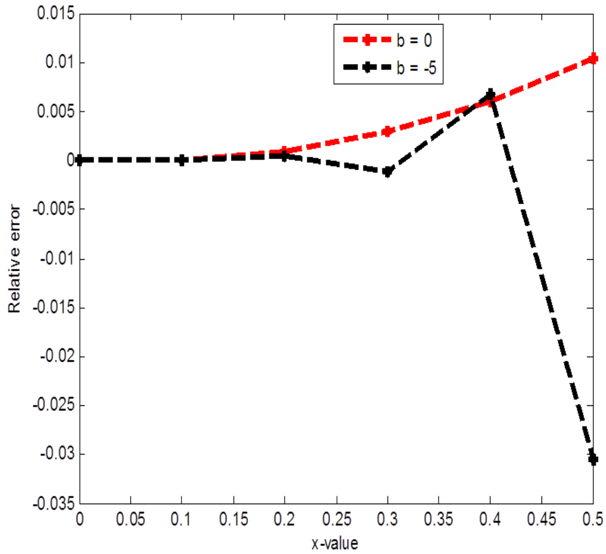

Figure 6: The comparative error analysis of the method for $b = 0$ and $b = -5$

The bound for the local truncation error when the derived two-step explicit method with $b = 0$ is applied to the initial value problem (17) is presented as follows. From (17), the first order differential equation is given by

$$y' = 4xy^{\frac{1}{2}} \tag{19}$$

In order to find a bound for the local truncation error when b = 0, we write that

$$y^{p+1}(x) = y^{2+1}(x) = y^{3}(x) \tag{20}$$

Where p is the order of the explicit linear multistep method for $b = 0$. We now differentiate the differential equation totally to obtain

$$y^{2}(x) = 4y^{\frac{1}{2}} + 8x^{2} \tag{21}$$

$$y^{3}(x) = 4xy^{\frac{1}{2}}.2y^{\frac{-1}{2}} + 16x = 8x + 16x = 24x \tag{22}$$

Hence, the bound for the magnitude of the local truncation error at any step within the range of integration say [*c*, *d*]

$$\left|L(y(x_n);h)\right| \leq h^{p+1}GY \tag{23}$$

where

$$Y = \max_{x\in[c,d]}\left|y^{p+1}(x)\right| \tag{24}$$

Since $p = 2$, then

$$Y = \max_{x\in[c,d]}\left|y^{3}(x)\right| = \max_{x\in[c,d]}\left|24x\right| = 24d \tag{25}$$

Where d is the right-hand end of the range of integration under consideration. The bound on the local truncation error is

$$\left|L(y(x_n);h)\right| \leq h^{3}G24d \tag{26}$$

But the influence function is obtained as

$$G = \frac{1}{12}\left|5+b\right| \tag{27}$$

Using the free parameter $b = 0$, (27) becomes

$$G = \frac{5}{12} \tag{28}$$

Combining (26) and (28), one gets

$$\left|L(y(x_n);h)\right| \leq h^{3}\left(\frac{5}{12}\right)24d = 10dh^{3} \tag{29}$$

The behavior of the bound for different values of the step length h is displayed in the Table 1 below.

Table 1: The bound of the local truncation error with different values of the step length h and $d = 2$.

Step Length, h	$L(y(x_n).h)$
0.1	0.020
0.01	0.000020
0.001	0.000000020
0.0001	0.000000000020
0.00001	0.00000000000002

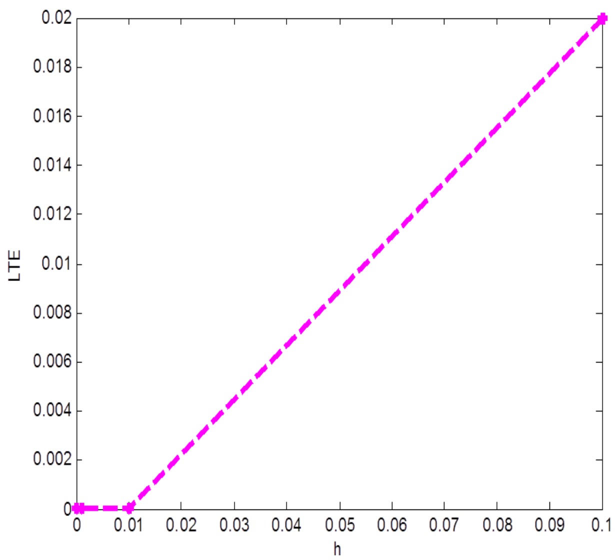

Figure 7: The effect of step length on the bound of the local truncation error with d = 2.

V. RESULTS AND DISCUSSION

The results obtained via the explicit linear two-step method for $b = 0$ and $b = -5$ are displayed in Figures 1 and 3, respectively. The errors obtained via the explicit linear two-step method for $b = 0$ and $b = -5$ are displayed in Figures 2 and 4, respectively. It is observed from Figure 5 that the results obtained for $b = 0$ via the explicit linear two-step method agree with the exact solution. It is clearly seen from Figure 6 that the method is zero stable for $b = 0$. It is also observed from Figure 6 that the method is zero unstable in the case of $b = -5$. It is observed from the Table 1 and Figure 7 that the smaller the step length, the smaller the bound for the local truncation error.

VI. CONCLUSION AND FUTURE SCOPE

In this work, an explicit linear two-step method of maximal order with one free parameter has been used for the solution of the IVP of the first order ODEs. The effect of the step length on the bound of the local truncation error has been investigated. The methodology can be extended for the solution of second order ordinary differential equations emanated from the real life situations with points of catastrophe by varying the free parameter.

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