Anti-magic labeling for Boolean graph of path $BG(P_n)$, $(n \geq 4)$

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Available online at: www.isroset.org
Accepted 12/Aug/2018, Online 30/Aug/2018

Abstract—A graph $G$ is anti-magic if there is a labelling of $G$ is a one-to-one mapping taking the edges onto 1, 2, ..., $|E|$ such that the sum of the labels assigned to edges incident to distinct vertices are different. A conjecture of Hartsfield and Ringel states that every connected graph different from $K_2$ is anti-magic. Our main result validates this conjecture for Boolean graph of path $P_n$ $(n \geq 4)$.

Keywords — Boolean graph $BG(G)$ Anti-magic Labeling

I. INTRODUCTION

Suppose $G(V,E)$ is a graph and let $E_G(v)$ be the set of edges of $G$ incident to $v$, for each vertex $v$ of $G$. We shall write $E(v)$ for $E_G(v)$. Let $f : E \rightarrow \{1, 2, \ldots, |E|\}$ be a bijective mapping. The vertex-sum $\varphi_f(v)$ at $v$ is defined as $\varphi_f(v) = \sum_{e \in E(v)} f(e)$. For any two distinct vertices $u,v$ of $G$, $\varphi_f(v) \neq \varphi_f(u)$ gives an anti-magic labeling of $G$. A graph $G$ is called anti-magic if $G$ has an anti-magic labeling.

The problem of anti-magic labeling of graphs was introduced by Hartsfield and Ringel [4]. They conjectured that all graphs with no single edge component are anti-magic. Graph Labeling has many applications in coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing, and data base management.

II. CONJECTURE 1

[4] Every connected graph different from $K_2$ is anti-magic. This conjecture is still open. Interestingly, the graph $K_2$ can be regarded as a tree on two vertices. Thus, if we restrict ourselves to trees, the above conjecture holds. Hartsfield and Ringel proved that paths, cycles and complete graph $K_n$, $(n \geq 3)$ are anti-magic. Recently, Alon et al. [1] have proved that the conjecture is true for some classes of dense graphs. They have shown that all dense graphs with $(n \geq 4)$ vertices and minimum degree $\Omega(\log n)$ are anti-magic. They also proved that if $G$ is a graph with $(n \geq 4)$ vertices and the maximum degree $\Delta(G) \geq 4n - 2$, then $G$ is anti-magic and all complete bipartite graphs except $K_2$ are anti-magic. Anti-magic labeling of the Cartesian product of graphs was studied in [7]; if $G$ is a regular anti-magic graph then for any graph $H$, the Cartesian product $H \times G$ is anti-magic. It was proved in [4] that 2-regular graphs are anti-magic and proved in [6] that 3-regular graphs are anti-magic. As a consequence, if $G$ is 2-regular or 3-regular then for any graph $H$, $H \times G$ is anti-magic. In this paper, we extend anti-magic labeling to Boolean Graph of path.

III. DEFINITION

Boolean graph $BG(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $BG(G)$ are adjacent if and only if they correspond to two adjacent vertices of $G$ or to a vertex and non-incident edge of $G$.

IV. THEOREM

The Boolean graph of path $BG(P_n)$, $(n \geq 4)$ is anti-magic.

Proof: Let $P_n$ be the path with vertices $v_1, v_2, v_3, \ldots, v_n$. By the definition of Boolean graph $BG(P_n)$ the vertex set is given by $V(BG(P_n)) = \{v_i : 1 \leq i \leq n\}$ and the edge set is given by $E(BG(P_n)) = \{u_j u_{j+1} : 1 \leq i \leq n - 1\}$.

We discuss Boolean graph of path in two cases.

Case (a): $n \equiv 1 \pmod{2}$

Label the vertices of $BG(P_n)$ using the function $f : E \rightarrow N$ as follows:

$f(v_i v_{i+1}) = i; \hspace{1cm} i = 1, 2, \ldots, n.$
$f(u_j u_{j+1}) = 2n-j; \hspace{1cm} j = 1, 2, \ldots, n-1.$
f(v_i, u_j) = (n-1) (i+1) + j if i < j, where 1 ≤ i ≤ n + 1 & 2 ≤ j ≤ n.
f(v_i, u_j) = (n-1) (i-1) + n + j if i > j, where j = 1, 2, ..., n & j < i ≤ n+1.
The induced function f*: V → N such that
f*(v_i) = Σ_{u_j ∈ E}(v_i) f(v_i, u_j)

We consider the case when labels of vertices are distinct.

Subcase (i): when i ≠ 1 and j ≠ i.

f*(v_i) = f(v_i, v_{i+1}) + Σ_{j=2}^{n} f(v_i, u_j)
= f(v_i, v_{i+1}) + Σ_{j=2}^{n} [(n-1)(i+1) + j]

Subcase (ii): When i = 2, 3, ..., n

f*(v_i) = f(v_{i-1}, v_i) + Σ_{j=i+1}^{n} f(v_i, u_j)
= i+1 - 1 + Σ_{j=1}^{i-2} f(v_i, u_j) + Σ_{j=i+1}^{n} f(v_i, u_j)
= 2i - 1 + Σ_{j=1}^{i-2} [(n-1)(i-1) + n + j] + Σ_{j=i+1}^{n} [(n-1)(i+1) + j]
= 2i - 1 + (i-2) [(n-1)(i-1) + n] + \frac{(i-2)(i-1)}{2} +

Subcase (iii): When i = n + 1 and j ≠ i.

f*(v_i) = f(v_{i-1}, v_i) + Σ_{j=1}^{n} f(v_i, u_j)
= f(v_{i-1}, v_i) + Σ_{j=1}^{n} [(n-1)(i-1) + j]

Subcase (iv): When i = 1 and j > i.

f*(u_j) = f(u_{i+1}, u_j) + Σ_{i=j+2}^{n} f(v_i, u_j)
= 2n - j + Σ_{i=j+2}^{n} [(n-1)(i-1) + n + j]
\[ f^*(u_i) = (2n-1) + \sum_{i=3}^{n+1} [(n-1)i - n + 1 + n + 1] \]
\[ = (2n-1) + \sum_{i=3}^{n+1} [(n-1)i + 2] \]
\[ = (2n-1) + (n-1)2 + (n-1) \]
\[ \frac{(n+1)(n+2) - 2.3}{2} \]
\[ = 2n-1 + 2n-2 + \frac{(n-1)(n^2 + 3n + 2)}{2} - 3(n-1) \]
\[ = \frac{1}{2} [8n-6 + n^3 + 3n^2 + 2n - n^2 - 3n - 2 - 6n + 6] \]
\[ f^*(u_i) = \frac{1}{2} [n^3 + 2n^2 + n - 2] \]

**Subcase (v): When \( j = 2, 3, \ldots, n-1 \)**

\[ f^*(u_i) = f(u_j, u_i) + f(u_j, u_{j+1}) + \sum_{i=j}^{n+1} f(v_j, u_i) \]
\[ = (2n-j + 1) + (2n-j) + \sum_{i<j} (n-1)i + \sum_{i=j+2}^{n+1} [(n-1)(-1) + n + j] \]
\[ = 4n-2j + 1 + \sum_{i=1}^{j-1} [(n-1)(i+1) + j] + \sum_{i=1}^{n+1} [(n-1)(i-1) + n + j] \]
\[ = 4n-2j + 1 + \frac{(n-1)(j-1)j}{2} + (j-1)(n-1)+j \]
\[ \frac{(n+1)(n+2) - (j+1)(j+2)}{2} + (n-j)(1+j) \]
\[ f^*(u_i) = \frac{1}{2} [n^3 + 2n^2 + 5n + 4 - 6j] \]

**Subcase (vi): When \( j = n \) and \( i < j \)**

\[ f^*(u_i) = f(u_j, u_i) + \sum_{i=j}^{n+1} f(v_i, u_j) \]
\[ = 2n-j + 1 + \sum_{i=1}^{j-1} [(n-1)(i+1) + j] \]
\[ f^*(u_i) = \frac{1}{2} [n^3 + 2n^2 + 5n + 4 - 6j] \]

**Subcase (vii): When \( i = 1 \) and \( j > i \)**

\[ f^*(v_i) = f(v_i, v_{i+1}) + \sum_{j=i}^{n} f(v_j, u_i) \]
\[ = 2n-i+1 + \sum_{j=1}^{n} [(n-1)(i+1) + j] \]
\[ f^*(v_i) = \frac{1}{2} [n^3 - 3n] \]

**Subcase (VIII): When \( i = 2, \ldots, n-1 \)**

\[ f^*(v_i) = f(v_{i-1}, v_i) + f(v_i, v_{i+1}) + \sum_{j=i+1}^{n} f(v_j, u_i) \]
\[ f^*(v_i) = f(v_{ij}) + \sum_{j=1}^{n} f(v_{ij}) \]
\[ = 2n - (n-1) + 1 + \sum_{j=1}^{n} (n-1)(i-1) + n + j \]
\[ = 2n - (n-1) + 1 + \sum_{j=1}^{n} (n-1)(i-1) + n + j \]
\[ = 1 + \sum_{j=1}^{n} (n-1)n + n + j \]
\[ = 1 + \sum_{j=1}^{n} (n^2 + j) \]
\[ = 1 + (n-1)n^2 + \frac{(n-1)n}{2} \]
\[ f^*(v_{n+1}) = \frac{1}{2} [2n^3 - n^2 - n + 2] \]

We consider the case when the labels of edges are distinct.

**Subcase (x): when \( j = 1 \) and \( i > j \)

\[ f^*(u_i) = f(u_{ij}) + \sum_{j=1}^{n} f(v_{ij}) \]
\[ = 2(n-j) + \sum_{i=1}^{n} [(n-1)(i-1) + n + j] \]
\[ f^*(u_i) = 2(n-1) + \sum_{i=1}^{n} [(n-1)(i-1) + n + j] \]
\[ = 2(n-1) + \sum_{i=1}^{n} [(n-1)(i-1) + n + j] \]
\[ = 2(n-1) + \sum_{i=1}^{n} [(n-1)(i-1) + n + j] \]
\[ = 2(n-1) + \sum_{i=1}^{n} [(n-1)(i-1) + n + j] \]
\[ = 2(n-1) + \sum_{i=1}^{n} [(n+1)(n+2) - \frac{2}{2} + \frac{2}{2}] + (n-1) \]
\[ = \frac{1}{2} [n^3 + 2n^2 + 5n + 6 - 10j] \]

**Subcase (xi): when \( j = 2, 3, ..., n-1 \)

\[ f^*(u_i) = f(u_{ij}) + \sum_{j=1}^{n} f(v_{ij}) \]
\[ = 2(n-j) + \sum_{i=1}^{n} [(n-1)(i-1) + n + j] \]
\[ f^*(u_i) = 2(n-1) + \sum_{i=1}^{n} [(n-1)(i-1) + n + j] \]
\[ = 2(n-1) + \sum_{i=1}^{n} [(n-1)(i-1) + n + j] \]
\[ = 2(n-1) + \sum_{i=1}^{n} [(n+1)(n+2) - \frac{2}{2} + \frac{2}{2}] + (n-1) \]
\[ = \frac{1}{2} [n^3 + 2n^2 + 5n + 6 - 10j] \]
\[ = 2(n-j) + 2 + \sum_{i=1}^{j-1}[(n-1)(i+1) + j] \]

\[ f^*(u_n) = 2(n-n) + 2 + \sum_{i=1}^{n-1}[(n-1)i + n - 1 + n] \]

\[ = 2 + (n-1) \frac{(n-1)n}{2} + (n-1)(2n-1) \]

\[ f^*(u_n) = \frac{1}{2} [n^3 + 2n^2 - 5n + 6] \]

Hence in all the above cases the labeling of all the vertices and the edges of the Boolean graph of path is anti-magic. 
\[ \therefore \text{BG}(P_n) \text{ is anti-magic.} \]

V. CONCLUSION

Finally we conclude that the anti-magic labeling to Boolean Graph of path is anti-magic.

REFERENCES


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