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Hamiltonian Cycle and Hamiltonian Path Decomposition of Fan Graphs

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Abstract - A path in a graph G that contains every vertex of G is called Hamiltonian path. A cycle in a graph G that contains every vertex of G is called a Hamiltonian cycle of G. A Hamiltonian graph is a graph that contains a Hamiltonian cycle. A graph is uniquely Hamiltonian if it contains exactly one Hamiltonian cycle. A decomposition of a graph G into Hamiltonian cycles will be called Hamiltonian decomposition of G. In this paper, we investigate Hamiltonian path decomposition of fan graph. Using this concept, we also investigate Hamiltonian decomposition of double fan graph.

Key words: *Hamiltonian cycle, Hamiltonian graph, uniquely Hamiltonian, Hamiltonian decomposition, fan graph.*

I. INTRODUCTION

Graph theory is proved to be tremendously useful in modeling the essential features of systems with finite components. Graphical models are used to represent telephone network, railway network, communication problems, traffic network etc. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and the relations by edges. The concept of decomposition of graphs in Hamiltonian cycles, Hamiltonian path decomposition of regular graphs was introduced by Klas Markstrom[1]. Decomposition of complete graphs into Hamilton cycles was discussed in [2] and J.C. Bermond [3] deals about Hamiltonian Decompositions of Graphs, Directed Graphs and Hypergraphs.

A path in a graph G that contains every vertex of G is called **Hamiltonian path**. A cycle in a graph G that contains every vertex of G is called a **Hamiltonian cycle** of G. A **Hamiltonian graph** is a graph that contains a Hamiltonian cycle. A graph is **uniquely Hamiltonian** if it contains exactly one Hamiltonian cycle. A decomposition of a graph G into Hamiltonian cycles will be called **Hamiltonian decomposition** of G.

Fan graph, denoted by $\boldsymbol{f_n}$**,** is the graph obtained from the path with *n* vertices P_n, where $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_i v_{i+1}: 1 \le i \le n-1\}$ by adding a new vertex *z* joining each v_i, $1 \le i \le n$ with *z*.

Double fan graph, denoted by $\boldsymbol{df_n}$, is the graph obtained from the fan f_n by adding a new vertex z_1 joining each v_i, $1 \le i \le n$ with z_1.

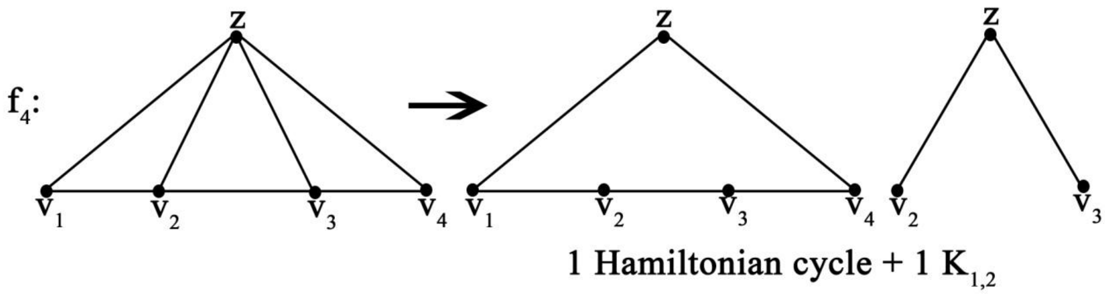
Section I contains the introduction of Hamiltonian decomposition of graphs and related topics, Section II contains Hamiltonian cycle decomposition of fan graphs, Section III contains Hamiltonian path decomposition of fan graphs, Section IV contains Hamiltonian decomposition of double fan graphs , Section V concludes research work.

II. HAMILTONIAN CYCLE DECOMPOSITION OF FAN GRAPH $\boldsymbol{f_n}$

Theorem: 2.1
The fan graph f_n, $n \ge 3$ is uniquely Hamiltonian and the rest of the edges form a star $K_{1,n-2}$.

Proof:
Let f_n , $n \geq 3$ be the fan graph with vertex set $V:\{v_1, v_2, \dots, v_n, z\}$. By definition, we have a path P_n with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$. By joining the end vertices of path P_n to the new vertex z, we get a spanning cycle. This is the only spanning cycle, for, the internal vertices of path P_n joining the new vertex z forms a star $K_{1,n-2}$. Therefore, the fan graph f_n , $n \geq 3$ is uniquely Hamiltonian.

Example: 2.2



III. HAMILTONIAN PATH DECOMPOSITION OF FAN GRAPH f_{n+2}

Theorem: 3.1

The fan graph f_{n+2} can be decomposed into $4n + 2$ Hamiltonian paths.

Proof:

Let f_{n+2} be the fan graph with vertex set $V:\{z, v_1, v_2, \dots, v_{n+1}, v_{n+2}\}$.

Consider the vertex z. We construct a path $P_{n+2}: v_1 v_2 \dots v_{n+1} v_{n+2}$ excluding the vertex z. Connect the vertex z to the vertex v_1 of P_{n+2}, we get a spanning path $Q_1: z v_1 v_2 \dots v_{n+1} v_{n+2}$. Also, connect the vertex z to the vertex v_{n+2} of P_{n+2}, we get a spanning path $Q_2: v_1 v_2 \dots v_{n+1} v_{n+2} z$. Therefore, considering the vertex z, we get two spanning paths.

Consider the vertex v_1. We construct $n + 2$ paths excluding the vertex v_1.

$R_1: z v_2 v_3 v_4 \dots v_n v_{n+1} v_{n+2}$
$R_2: v_2 z v_3 v_4 \dots v_n v_{n+1} v_{n+2}$
$R_3: v_2 v_3 z v_4 \dots v_n v_{n+1} v_{n+2}$
. . .
. . .
. . .
$R_n: v_2 v_3 v_4 \dots v_n z v_{n+1} v_{n+2}$
$R_{n+1}: v_2 v_3 v_4 \dots v_n v_{n+1} z v_{n+2}$
$R_{n+2}: v_2 v_3 v_4 \dots v_n v_{n+1} v_{n+2} z$

Connect the vertex v_1 to the starting vertex of each paths R_i, $1 \leq i \leq n + 2$, we get $n + 2$ spanning paths. But the spanning path $R'_{n+2}: v_1 v_2 v_3 v_4 \dots v_n v_{n+1} v_{n+2} z$ is same as $Q_2: v_1 v_2 \dots v_{n+1} v_{n+2} z$. So we get $n + 1$ spanning paths.

Also, we construct n paths excluding the vertex v_1.

$S_1: z v_{n+2} v_{n+1} v_n \dots v_5 v_4 v_3 v_2$
$S_2: v_2 z v_{n+2} v_{n+1} v_n \dots v_5 v_4 v_3$
$S_3: v_2 v_3 z v_{n+2} v_{n+1} v_n \dots v_5 v_4$
. . .
. . .
. . .
$S_{n-2}: v_2 v_3 v_4 \dots v_{n-2} z v_{n+2} v_{n+1} v_n v_{n-1}$
$S_{n-1}: v_2 v_3 v_4 \dots v_{n-1} z v_{n+2} v_{n+1} v_n$
$S_n: v_2 v_3 v_4 \dots v_{n-1} v_n z v_{n+2} v_{n+1}$

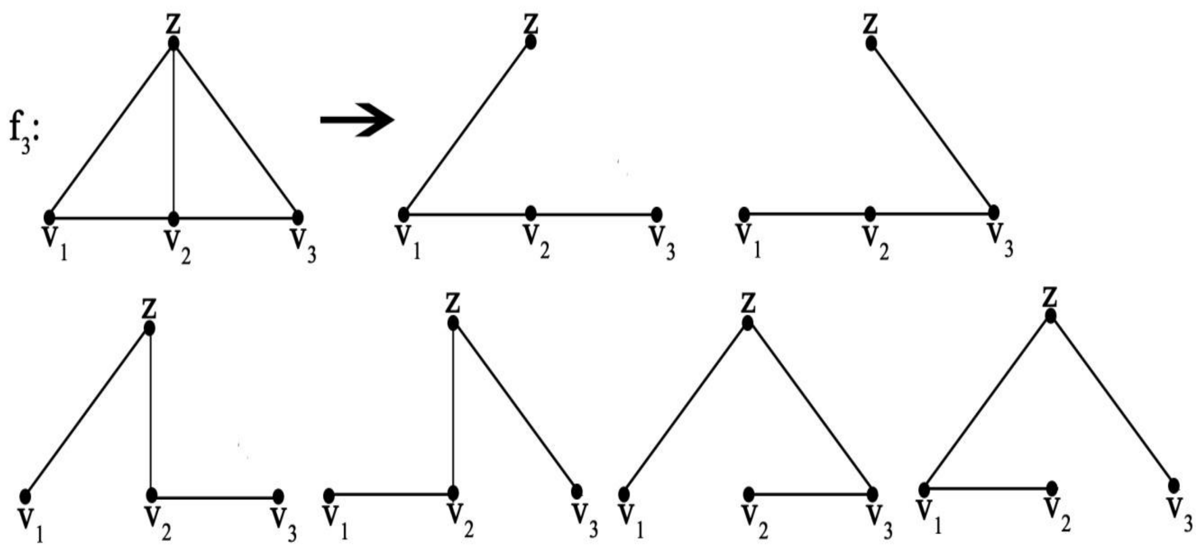
Connect the vertex v_1 to the starting vertex of each paths S_i, $1 \leq i \leq n$, we get n spanning paths. Therefore, considering the vertex v_1 we get $2n + 1$ spanning paths.

Proceeding like this we get a sequence of Hamiltonian paths $2, 2n + 1, 2, 2, \dots, 2, 1$.

The number of Hamiltonian paths in $f_{n+2} = (2 + 2n + 1) + (2 + 2 + \cdots + 2)(n - 1) times + 1$

$$= 2n + 3 + 2(n-1) + 1$$
$$= 2n + 4 + 2n - 2$$
$$= 4n + 2.$$

Example: 3.2



6 Hamiltonian paths

IV. HAMILTONIAN DECOMPOSITION OF DOUBLE FAN GRAPH df_{n+2}

Theorem: 4.1

The double fan graph df_{n+2} can be decomposed into $4n$ Hamiltonian cycles.

Proof:

Let df_{n+2} be the double fan graph with vertex set $V: \{z, v_1, v_2, \dots, v_{n+1}, v_{n+2}, z_1\}$ where $\{z, v_1, v_2, \dots, v_{n+1}, v_{n+2}\}$ be the vertex set of fan graph f_{n+2} and z_1 be the new vertex.

In fan graph f_{n+2}, consider the vertices of fan f_{n+2} with the condition that z is not the end vertices of the path and also connecting all vertices of fan f_{n+2}, we get $4n$ Hamiltonian paths.

In double fan graph df_{n+2} , consider the vertex v_1, we construct $n+1$ paths starting with the vertex v_1.

$Q_1: v_1 z v_2 v_3 v_4 \dots v_n v_{n+1} v_{n+2}$
$Q_2: v_1 v_2 z v_3 v_4 \dots v_n v_{n+1} v_{n+2}$
$Q_3: v_1 v_2 v_3 z v_4 \dots v_n v_{n+1} v_{n+2}$
. . .
. . .
. . .
$Q_n: v_1 v_2 v_3 v_4 \dots v_n z v_{n+1} v_{n+2}$
$Q_{n+1}: v_1 v_2 v_3 v_4 \dots v_n v_{n+1} z v_{n+2}$

Connect the new vertex z_1 to the end vertices of each paths Q_i, $1 \le i \le n+1$, we get $n+1$ spanning cycles.

Also, we construct n paths starting with the vertex v_1.

$R_1: v_1 z v_{n+2} v_{n+1} v_n \dots v_5 v_4 v_3 v_2$
$R_2: v_1 v_2 z v_{n+2} v_{n+1} v_n \dots v_5 v_4 v_3$
$R_3: v_1 v_2 v_3 z v_{n+2} v_{n+1} v_n \dots v_5 v_4$
. . .
. . .
. . .
$R_{n-2}: v_1 v_2 v_3 v_4 \dots v_{n-2} z v_{n+2} v_{n+1} v_n v_{n-1}$
$R_{n-1}: v_1 v_2 v_3 v_4 \dots v_{n-1} z v_{n+2} v_{n+1} v_n$
$R_n: v_1 v_2 v_3 v_4 \dots v_{n-1} v_n z v_{n+2} v_{n+1}$

Connect the new vertex z_1 to the end vertices of each paths R_i, $1 \le i \le n$, we get n spanning cycles. Therefore, considering the vertex v_1, we get $2n+1$ spanning cycles.

Consider the vertex v_2, we have two cases.

Case: (i) $n = 1$

Consider the vertex v_2. We construct two paths starting with the vertex v_2, $S_1: v_2 v_1 z v_3$ and $S_2: v_2 v_3 z v_1$ when $n = 1$.

Connect the new vertex z_1 to the end vertices of each paths S_i, $1 \le i \le 2$, we get two spanning cycles. But the spanning cycle $S_2': z_1 v_2 v_3 z v_1 z_1$ is same as $R_1': z_1 v_1 z v_3 v_2 z_1$ when $n = 1$. So we get one spanning cycle.

Case: (ii) $n \ge 2$

Consider the vertex v_2. We construct three paths starting with the vertex v_2.

$T_1: v_2 v_1 z v_3 v_4 \dots v_{n+1} v_{n+2}$
$T_2: v_2 v_1 z v_{n+2} v_{n+1} \dots v_4 v_3$
$T_3: v_2 v_3 v_4 \dots v_{n+1} v_{n+2} z v_1$

Connect the new vertex z_1 to the end vertices of each paths T_i, $1 \le i \le 3$, we get three spanning cycles. But the spanning cycle $T_3': z_1 v_2 v_3 v_4 \dots v_{n+1} v_{n+2} z v_1 z_1$ is same as $R_1': z_1 v_1 z v_{n+2} v_{n+1} v_n \dots v_5 v_4 v_3 v_2 z_1$ when $n \ge 2$. So we get two spanning cycles.

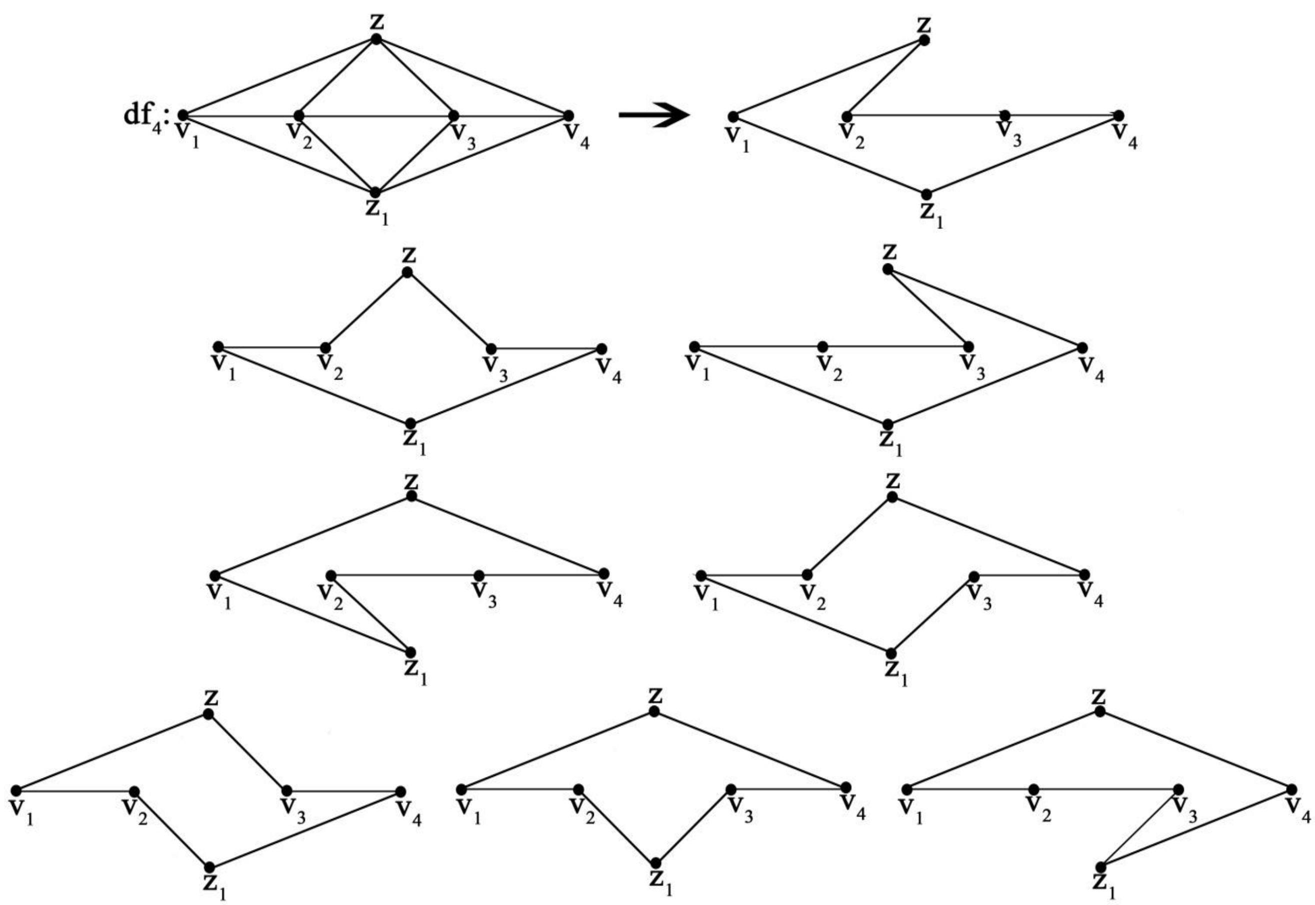
Therefore, considering the vertex v_2, we get $\begin{cases} 1 \; spanning\ cycle\ if\ n = 1 \\ 2 \; spanning\ cycles\ if\ n \ge 2 \end{cases}$.

Proceeding like this we get a sequence of Hamiltonian cycles $2n+1, 2, 2, \dots, 2, 1$.

The number of Hamiltonian cycles in $df_{n+2} = (2n+1) + (2+2+\cdots+2)(n-1) times + 1$

$$= 2n + 1 + 2(n-1) + 1$$
$$= 2n + 2 + 2n - 2$$
$$= 4n.$$

Example: 4.2



8 Hamiltonian cycles

V. CONCLUSION

We investigate Hamiltonian Cycle Decomposition of Fan Graphs and Hamiltonian Path Decomposition of Fan Graphs using the concept of Hamiltonian Decomposition. Also, we investigate Hamiltonian Decomposition of Double Fan Graphs using the concept of Hamiltonian Path Decomposition of Fan Graphs.

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