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# A Competitive Study on the Euler and Different Order Runge-Kutta Methods with Accuracy and Stability

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***Abstract***— Numerical methods in solving ordinary differential equations (ODEs) play an essential role in dealing with different mathematical, physical, and engineering problems. The accuracy and stability of the numerical methods are the critical factors for examining their performances in solving related problems. In this study, a comparison is made between some well-known methods, namely the Euler and different order Runge-Kutta (RK) methods, which are RK(2,2), RK(3,3), and RK(4,4) schemes, in terms of accuracy and stability. We have tested the numerical accuracy of the mentioned methods through the root mean square error (RMSE) value. Furthermore, the stability polynomials of the methods are determined, and their contour plot is made to analyze the methods' stability. We have got the ordered relation "Euler $\leq$ RK(2,2) $\leq$ RK(3,3) $\leq$ RK(4,4)" both for the accuracy and stability tests. According to the ordered relation, it is clear that the RK(4,4) method is the most accurate and stable scheme among the test methods.

***Keywords***—Euler method, Different order Runge-Kutta methods, Accuracy, Stability, Root mean square error, Stability polynomial

## I. INTRODUCTION

Numerical analysis is the union of the theoretical and computational investigation into the computer solution of mathematical problems. It involves the study of methods of computing numerical data. In many problems, this implies producing a sequence of approximations. Thus, the questions involve the rate of convergence, the accuracy (or even validity) of the answers, and the stability of a numerical method [1]. Primary areas of theoretical concern in numerical analysis are global or local error bounding, stability of methods, and rates of convergence of numerical methods [1]. The convergence of the methods is directly connected with the accuracy of the approximated solutions produced through them and their stability. The numerical analysis's role is to derive efficient numerical methods for obtaining approximate solutions of many mathematical equations and analyze the methods' accuracy and stability properties. Thus, numerical analysis plays a vitally important role in all branches of science and engineering. It is to be noted here that numerical ordinary differential equations (ODEs) is part of numerical analysis, which studies and analyses the numerical solutions of ODEs. ODEs occur in many scientific disciplines, such as mechanics, chemistry, biology, and economics. Also, sometimes the traveling wave equations are transformed into ODEs, and their solutions are obtained by solving these ODEs [2-4]. These ODEs can be solved by analytically and numerically. It can be mentioned here that many ODEs cannot be solved analytically, in which case we have to satisfy ourselves with some approximations to their solutions [5-8]. The methods studied here in numerical analysis can compute such an approximation. Let us suppose that we want approximate the solution of the differential equation

$$y' = f(t, y(t)), \qquad y(t_0) = y_0, \tag{1}$$

where $f$ is a function that maps $[t_0, \infty) \times R^d$ to $R^d$, and the initial condition $y_0 \in R^d$ is a given vector [9]. The above formulation (1) is called an initial value problem (IVP). The Picard-Lindelof theorem states that there is a unique solution if $f$ is Lipschitz continuous. However, the Euler and different order Runge-Kutta (RK) methods, namely, RK(2,2), RK(3,3), and RK(4,4) methods, are some well-known methods for solving IVPs with modern computers. But as mentioned above, the accuracy and stability of a numerical method should carefully be taken into consideration.

Thus, in our study, a comparison is made between the Euler and different order RK methods in terms of accuracy and stability. We have considered an IVP to test the methods' accuracy and analyze their stability. We have tested the numerical accuracy of the mentioned methods through the root mean square error (RMSE) value. The stability polynomials of the methods are determined for the IVP, and their contour plot is made to analyze the methods' stability.

The rest of the paper is organized as follows. Section II discusses the preliminaries. Section III presents the Algorithms of the considered test methods. Then, Section IV deals with the simulation outcomes. Finally, the conclusion and future scope are presented in Section V.

## II. PRELIMINARIES

In this section, some preliminaries and definitions are discussed as follows:

*IVP:* In an $n$ order differential equation, if the values of dependent variable $y(x)$ and its first $(n-1)$ order derivatives are specified at a particular point, say $x = x_0$ then such a differential equation is called an initial problem. The specified conditions at the particular point are called initial conditions. There are the form

$$y(x_0)=k_{0,}\ y'(x_0)=k_1,...,\ y^{(n-1)}(x_0)=k_{n-1}, \tag{2}$$

where $k_0, k_1,...,k_{n-1}$ are the specified constants.

*Accuracy:* Accuracy refers to how closely a computed, approximated, or measured value agrees with the true/exact value.

*Stability:* In mathematics, it is a condition in which a slight disturbance in a system does not produce too disrupting an effect on that system.

*RMSE:* The root of the mean of the squares of some given errors is known as the RMSE value. It is used to measure the accuracy of a method. Let $e_1, e_2, \ldots, e_n$ are some errors. Then,

$$rmse=\sqrt{\frac{e_1^2+e_2^2+\cdots+e_n^2}{n}}. \tag{3}$$

## III. ALGORITHMS

The detail of the IVP solvers, namely the Euler, RK(2,2), RK(3,3), and RK(4,4) methods, is skipped because it can be found in many well-reputed books, one can refer to [1, 10]. Instead, the algorithms of the mentioned IVP solvers are presented for writing a computer program with a suitable programming language. The algorithms are given as follows:

*Algorithm 1:* The algorithm for the Euler method is given below.

```
Step 1  INPUT:: endpoints a, b; number of
        approximations N; initial condition c.
Step 2  Set h = (b − a)/N;
            x = a;
            y = c;
        Output (x, y).
Step 3  For i = 1, 2, . . . , N do the followings:
            Set y = y + h*f (x, y);
                x = a + i*h.
            Output (x, y).
Step 4  STOP.
```

*Algorithm 2:* The algorithm for the RK(2,2) method is given below.

```
Step 1  INPUT:: endpoints a, b; number of
        approximations N; initial condition c.
Step 2  Set h = (b − a)/N;
            x = a;
            y = c;
        Output (x, y).
Step 3  For i = 1, 2, . . . , N do the followings:
            Set K1 = h*f (x, y);
                K2 = h*f (x + h, y + K1);
                y = y + (K1 + K2)/2;
                x = a + i*h.
            Output (x, y).
Step 4  STOP.
```

*Algorithm 3:* The algorithm for the RK(3,3) method is given below.

```
Step 1  INPUT:: endpoints a, b; number of
        approximations N; initial condition c.
Step 2  Set h = (b − a)/N;
            x = a;
            y = c;
        Output (x, y).
Step 3  For i = 1, 2, . . . , N do the followings:
            Set K1 = h*f (x, y);
                K2 = h*f (x + h/2, y + K1/2);
                K3 = h*f (x + h, y + 2*K2 – K1);
                y = y + (K1 + 4*K2 + K3)/6;
                x = a + i*h.
            Output (x, y).
Step 4  STOP.
```

*Algorithm 4:* The algorithm for the RK(4,4) method is given below.

```
Step 1  INPUT:: endpoints a, b; number of
        approximations N; initial condition c.
Step 2  Set h = (b − a)/N;
            x = a;
            y = c;
        Output (x, y).
Step 3  For i = 1, 2, . . . , N do the followings:
            Set K1 = h*f (x, y);
                K2 = h*f (x + h/2, y + K1/2);
                K3 = h*f (x + h/2, y + K2/2);
                K4 = h*f (x + h, y + K3);
                y = y+(K1+2*K2+2*K3+K4)/6;
                x = a + i*h.
            Output (x, y).
Step 4  STOP.
```

 

We have developed some MATLAB programs using the given algorithms for computer implementation. Then, the accuracy of the approximated solutions, obtained with the methods, and the methods' stability are compared.

## IV. RESULTS AND DISCUSSION

The aforementioned MATLAB programs are implemented on a computer with the Windows 8.1 Pro 64-bit operating system with eight gigabytes RAM and the Intel Core i5-4570 CPU @ 3.20 GHz hardware configuration. We have considered the following IVP for testing the accuracy of its approximated solutions obtained by the Euler, RK(2,2), RK(3,3), and RK(4,4) methods.

$$y' = -y, \text{ with } y(0) = 1 \text{ and } 0 \le x \le 1. \quad (4)$$

We have solved the IVP of Eq. (4) with the test methods mentioned with step size $h = 0.1$. The Solutions obtained by the test methods with the exact solutions and errors between them are presented in Table 1. The errors $E_{eui}$, $E_{rk(2,2)i}$, $E_{rk(3,3)i}$, and $E_{rk(4,4)i}$ between the exact solution $Y_i$ of the IVP and the approximated solutions $Y_{eui}$, $Y_{rk(2,2)i}$, $Y_{rk(3,3)i}$, and $Y_{rk(4,4)i}$ obtained by the Euler, RK(2,2) RK(3,3), and RK(4,4) methods, respectively, at step $i$ are given in Table 1 with the exact and approximated solutions.

Table 2. The RMSE values for the test methods

| Methods | RMSE values |
|---|---|
| Euler | 0.0148 |
| RK(2,2) | 5.0732e-04 |
| RK(3,3) | 1.2739e-05 |
| RK(4,4) | 2.5562e-07 |

From Table 2, it is found that the RMSE values of the Euler, RK(2,2), RK(3,3), and RK(4,4) methods are of order $10^{-2}$, $10^{-4}$, $10^{-5}$, and $10^{-7}$, respectively. It is to be noted here that the order of the RMSE value for a method indicates its accuracy level in obtaining solutions of a problem. The order of the RMSE value for the RK(4,4) method is the smallest, which is acceptable in scientific computing. According to the RMSE values, the following ordered relation can be made

$$Euler \le RK(2,2) \le RK(3,3) \le RK(4,4). \quad (5)$$

From the ordered relation represented by (5), it can be said that the first, second, and third most accurate approximated solutions of the IVP presented by Eq. (4) are obtained by the RK(4,4), RK(3,3), and RK(2,2) methods. On the other hand, the Euler method produced the least accurate ones. It is also clear that the higher the order of an IVP solver is, the higher the accuracy of the approximated solutions obtained by the IVP solver is.

Table 1. The approximated solutions obtained by the Euler, RK(2,2), RK(3,3), and RK(4,4) methods with exact solutions & errors between them

| Number of steps, i | $X_i$ | Exact solutions, $Y_i$ | Approximated solutions | | | | Errors | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $Y_{eui}$ | $Y_{rk(2,2)i}$ | $Y_{rk(3,3)i}$ | $Y_{rk(4,4)i}$ | $E_{eui}$ | $E_{rk(2,2)i}$ | $E_{rk(3,3)i}$ | $E_{rk(4,4)i}$ |
| 00 | 0.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 01 | 0.1000 | 0.9048 | 0.9000 | 0.9050 | 0.9048 | 0.9048 | 0.0048 | -0.0002 | 4.0847e-06 | -8.1964e-08 |
| 02 | 0.2000 | 0.8187 | 0.8100 | 0.8190 | 0.8187 | 0.8187 | 0.0087 | -0.0003 | 7.3919e-06 | -1.4833e-07 |
| 03 | 0.3000 | 0.7408 | 0.7290 | 0.7412 | 0.7408 | 0.7408 | 0.0118 | -0.0004 | 1.0032e-05 | -2.0132e-07 |
| 04 | 0.4000 | 0.6703 | 0.6561 | 0.6708 | 0.6703 | 0.6703 | 0.0142 | -0.0005 | 1.2104e-05 | -2.4288e-07 |
| 05 | 0.5000 | 0.6065 | 0.5904 | 0.6070 | 0.6065 | 0.6065 | 0.0160 | -0.0005 | 1.3690e-05 | -2.7471e-07 |
| 06 | 0.6000 | 0.5488 | 0.5315 | 0.5494 | 0.5488 | 0.5488 | 0.0174 | -0.0006 | 1.4865e-05 | -2.9828e-07 |
| 07 | 0.7000 | 0.4966 | 0.4783 | 0.4972 | 0.4966 | 0.4966 | 0.0183 | -0.0006 | 1.5691e-05 | -3.1488e-07 |
| 08 | 0.8000 | 0.4493 | 0.4305 | 0.4499 | 0.4493 | 0.4493 | 0.0189 | -0.0006 | 1.6227e-05 | -3.2562e-07 |
| 09 | 0.9000 | 0.4066 | 0.3874 | 0.4972 | 0.4066 | 0.4066 | 0.0191 | -0.0006 | 1.6518e-05 | -3.3146e-07 |
| 10 | 1.0000 | 0.3679 | 0.3487 | 0.3685 | 0.3679 | 0.3678 | 0.0192 | -0.0007 | 1.6607e-05 | -3.3324e-07 |

From Table 1, it is clear that the error for the RK(4,4) method is the least among the errors for the other test methods at each step. However, the RMSE value test can be an appropriate option to analyze errors with less effort. It is to be noted here that the accuracy of the approximated solutions obtained by any numerical method is inversely proportional to the RMSE value. Thus, the RMSE values for the test methods are calculated using the errors from Table 1 and presented in Table 2.

Moreover, to derive the stability polynomials of the test methods, we have considered the test equation

$$y' = \lambda y, \quad (6)$$

where $\lambda$ is a complex number. Using Eq. (6), the stability polynomials for the Euler, RK(2,2), RK(3,3), and RK(4,4) methods are determined, which can be represented as

$$Q_e(z) = 1 + z, \quad (7)$$

 

$$Q_{rk_{2,2}}(z) = 1 + z + \frac{z^2}{2}, \tag{8}$$

$$Q_{rk_{3,3}}(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6}, \tag{9}$$

$$Q_{rk_{4,4}}(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}, \tag{10}$$

respectively. Here $z = \lambda h$, and $h$ is the step size. The contour plots of the stability polynomials given by Eqs. (7)-(10) are presented in Fig. 1. It is clear from Fig. 1 that the contour plot of the polynomials for the RK(4,4), RK(3,3), and RK(2,2) methods occupy the first, second, and third-largest regions, respectively, where that for the Euler method occupies the smallest region.

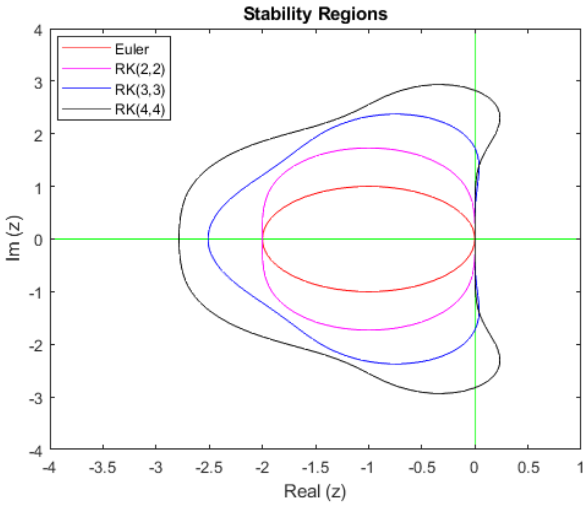


Figure 1. Stability regions of the Euler, RK(2,2), RK(3,3), and RK(4,4) methods.

It can be noted here that the larger the region is, the more stable the corresponding method is. Hence, in terms of stability, we can derive the following ordered relation as

$$Euler \leq RK(2,2) \leq RK(3,3) \leq RK(4,4). \tag{11}$$

From the ordered relation represented by (11), it is found that the RK(4,4) method is the most stable one among the test methods. In contrast, the Euler method is the least stable one.

In this work, we have analyzed the accuracy and stability, two essential criteria for studying numerical methods, of the Euler, RK(2,2), RK(3,3), and RK(4,4) methods. Two ordered relations, represented by (5) and (11), have been derived that reflect how accurate and stable the test methods are, respectively, for solving the considered IVP.

## V. CONCLUSION AND FUTURE SCOPE

Some well-known numerical methods used in solving IVPs, namely, the Euler, RK(2,2), RK(3,3), and RK(4,4) methods, with the ability to be used in computers, were considered for examining their accuracy and stability. The accuracy was judged in terms of the RMSE value, and the stability was investigated by analyzing the stability regions of the test methods. We found the RK(4,4) method as the most accurate and stable method among the test methods.

However, it is well known that the accuracy of the Taylor series method is proportional to the number of considered higher-order terms. Furthermore, the Euler and different order RK methods are derived from the Tylor series method. Thus, there is a scope to examine whether "the higher order RK methods always have more accuracy and stability over the lower order RK methods or not." This study can help analyze the higher-order RK methods having an order greater than four.

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