Generalized Common Fixed Point Theorem for Four Self Mappings in b-Metric Space

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Available online at: www.isroset.org

Accepted 27/Jun/2018, Online 30/Aug/2018

Abstract- In this paper we have tried to prove a common fixed point theorem of four self-mappings satisfying a generalized contractive condition in b-metric space. Our result extend several comparable pre-existing results in literature. We have also given an example that shows the applicability and validity of our result.

Keywords- Fixed point; Common Fixed point; Cauchy Sequence; compatible mapping; b- Metric space

2000 classification: 47H10, 54H25

I. INTRODUCTION AND PRELIMINARIES

The idea of b-metric space was introduced by Bourbaki [6] and Bakhtin[3] in 1989. Czerwik[8] give an axiom which was weaker than triangle inequality and formally defined a b-metric space with a view of generalization the Banach contraction mapping theorem. Since then several research paper have dealt with fixed point theory for single and multi-valued mapping in b- metric spaces .Pacurar [12] extended the fixed point theorem in b-metric space. Many authors in their work have used the b-metric spaces in which b- metric function is continuous , but the techniques used here can be employed in the setup of discontinuous –b metric spaces.in this paper we focused on Ćirić and Hardy-Rogers type contractions and present a common fixed point in b-metric spaces.

Our result extended and improve the result of J.R. Roshan , N.Shobkolaei, S.Sedghi and M. Abbas [10] and several comparable result that exist in literature , Following definitions and result will be needed in the sequel.

Definition 1.1: Let X be a non-empty set and $s \geq 1$ be a given real no., function $d: X \times X \rightarrow \mathbb{R}^+$ is a b-metric iff for each $x, y, z \in X$ if following conditions are hold ;

- (1) $d(x, y) = 0$ iff $x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, z) \leq s [d(x, y) + d(y, z)]$

A pair (X, d) is called a b-metric space.

Definition 1.2: Let (X, d) be a b-metric space .then a sequence $\{x_n\}$ in X is called

- (i) Convergent if and only if there exist $x \in X$ such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
- (ii) Cauchy if and only if $d(x_n, x_m) \rightarrow 0$ as $m, n \rightarrow \infty$.
- (iii) The b-metric space (X, d) is complete if every Cauchy sequence is convergent.

Definition 1.3: (X, d) be a b-metric space. A pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n \rightarrow \infty} d(fg x_n, g f x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = t$ for some $t \in X$

Lemma 1.4: Let (X, d) be a b-metric space with $k \geq 1$, suppose that $\{x_n\}$ and $\{y_n\}$ are b -convergent such that to x and y respectively .then we have

$$\frac{1}{k^2} d(x, y) \leq \lim_{n \rightarrow \infty} \inf d(x_n, y_n) \leq \lim_{n \rightarrow \infty} \sup d(x_n, y_n) \leq k^2 d(x, y)$$

In Particular, if $x=y$, then we have $\lim_{n \rightarrow \infty} d(x_n, y_n)=0$ moreover for each $z \in X$ we have

$$\frac{1}{k} d(x, z) \leq \lim_{n \rightarrow \infty} \inf d(x_n, z) \leq \lim_{n \rightarrow \infty} \sup d(x_n, z) \leq k d(x, z)$$

Lemma 1.5: Let (X, d) be a b-metric space . If there exist two sequence $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \rightarrow \infty} d(x_n, y_n)=0$. Wherever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} x_n=t$ for some $t \in X$ then $\lim_{n \rightarrow \infty} y_n=t$.

II. MAIN RESULT

Theorem (2.1) : suppose that F, G, S and T are self-mapping on a complete b-metric space (X, d) such that $F(X) \subseteq T(X)$, $G(X) \subseteq S(X)$, if

$$(2.1) d(Fx, Gy) \leq \frac{a_1}{k^4} [a_2 \max\{d(Sx, Ty), d(Fx, Sx), d(Gy, Ty), \frac{1}{2}(d(Sx, Gy) + d(Fx, Ty))\} + (1 - a_2)\{\max(d^2(Sx, Ty), d(Fx, Sx)d(Gy, Ty), d(Sx, Gy).d(Fx, Ty), \frac{d(Fx, Sx)d(Fx, Sy)}{2}, \frac{d(Sy, Gy)d(Sx, Gy)}{2})\}^{\frac{1}{2}}].$$

Where $a_1 \in (0, 1)$ and $0 \leq a_2 \leq 1$

Holds for each $x, y \in X$ with $0 < a_1 < 1$ then F, G, S and T have a unique common fixed point in X provided that S and T are continuous and pairs (F, S) and (G, T) are compatible.

Proof: Let $x_0 \in X$ as $F(X) \subseteq T(X) \exists x_1 \in X$ s.t $F(x_0) = Tx_1$. since $Gx_1 \in S(X)$, we can chose $x_2 \in X$ such that $Gx_1 = Sx_2$ in general x_{2n+1} and x_{2n+2} are chosen in X such that $Fx_{2n} = Tx_{2n+1}$ and $Gx_{2n+1} = Sx_{2n+2}$ define a sequence $\{y_n\}$ in X such that $y_{2n} = Fx_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Gx_{2n+1} = Sx_{2n+2}$ for all $n \geq 0$ Now we show that $\{y_n\}$ is a Cauchy sequence, consider

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Fx_{2n}, Gx_{2n+1}) \leq \frac{a_1}{k^4} [a_2 \max\{d(Sx_{2n}, Tx_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \\ &\frac{1}{2}(d(Sx_{2n}, Gx_{2n+1}) + d(Fx_{2n}, Tx_{2n+1}))\} + (1 - a_2)\{\max(d^2(Sx_{2n}, Tx_{2n+1}), d(Fx_{2n}, Sx_{2n})d(Gx_{2n+1}, Tx_{2n+1}), \\ &d(Sx_{2n}, Gx_{2n+1}).d(Fx_{2n}, Tx_{2n+1}), \frac{d(Fx_{2n}, Sx_{2n})d(Fx_{2n}, Sx_{2n+1})}{2}, \frac{d(Sx_{2n+1}, Gx_{2n+1})d(Sx_{2n}, Gx_{2n+1})}{2})\}^{\frac{1}{2}}] \\ &= \frac{a_1}{k^4} [a_2 \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \frac{1}{2}(d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n}))\} + \\ &(1 - a_2)\{\max(d^2(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}).d(y_{2n}, y_{2n}), \\ &\frac{d(y_{2n}, y_{2n-1})d(y_{2n}, y_{2n})}{2}, \frac{d(y_{2n}, y_{2n+1})d(y_{2n-1}, y_{2n+1})}{2})\}^{\frac{1}{2}}] \\ &= \frac{a_1}{k^4} [a_2 \max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), \frac{1}{2}(d(y_{2n-1}, y_{2n+1}))\} + \\ &(1 - a_2)\{\max(d^2(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n+1}, y_{2n}), \frac{d(y_{2n}, y_{2n+1})d(y_{2n-1}, y_{2n+1})}{2})\}^{\frac{1}{2}}] \end{aligned}$$

$$\leq \frac{a_1}{k^4} [a_2 \max\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), \frac{k}{2}(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}))\} + (1-a_2)\{\max(d^2(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n})d(y_{2n+1}, y_{2n}), \frac{d(y_{2n}, y_{2n+1})k(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})))}{2}\}^{1/2}] \quad \text{if } d$$

$(y_{2n}, y_{2n+1}) > d(y_{2n-1}, y_{2n})$, for some n then from above inequality we have

$$d(y_{2n}, y_{2n+1}) < \frac{a_1}{k^4} [a_2 kd(y_{2n}, y_{2n+1}) + (1-a_2)\{kd^2(y_{2n}, y_{2n+1})\}^{1/2}]$$

$$d(y_{2n}, y_{2n+1}) < \frac{a_1}{k^4} [a_2 kd(y_{2n}, y_{2n+1}) + (1-a_2)\{kd(y_{2n}, y_{2n+1})\}]$$

$$= \frac{a_1}{k^4} [kd(a_2 + (1-a_2))d(y_{2n}, y_{2n+1})]$$

$$= \frac{a_1}{k^3} d(y_{2n}, y_{2n+1})$$

$$d(y_{2n}, y_{2n+1}) < \frac{a_1}{k^3} d(y_{2n}, y_{2n+1})$$

Which is a contradiction;

Hence $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$ for all $n \in \mathbb{N}$ also by the above inequality we have

$$(2.2) d(y_{2n}, y_{2n+1}) \leq \frac{a_1}{k^3} d(y_{2n-1}, y_{2n})$$

$$(2.3) d(y_{2n-1}, y_{2n}) \leq \frac{a_1}{k^3} d(y_{2n-2}, y_{2n-1})$$

From (2.2) and (2.3) we have

$$d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2})$$

Where $\lambda = \frac{a_1}{k^3} < 1$ and $n \geq 2$, hence for all $n \geq 2$ we obtain

$$d(y_n, y_{n-1}) \leq \dots \leq \lambda^{m-1} d(y_1, y_0) \quad (2.4), \text{ so for all } n > m \text{ we have}$$

$$d(y_n, y_m) \leq kd(y_m, y_{m+1}) + k^2 d(y_{m+1}, y_{m+2}) + \dots + k^{n-m-1} d(y_{n-1}, y_n).$$

from (2.4) we have

$$\begin{aligned}
d(y_n, y_m) &\leq (k\lambda^m, k^2\lambda^{m+1} + \dots + k^{n-m-1}\lambda^{n-1}) d(y_1, y_0) \\
&= k\lambda^m (1 + k\lambda + (k\lambda)^2 + \dots) d(y_1, y_0) \\
&\leq \frac{k\lambda^m}{1 - k\lambda} d(y_1, y_0)
\end{aligned}$$

On taking as $m, n \rightarrow \infty$ we have $d(y_n, y_m) \rightarrow 0$ as $k\lambda < 1$, therefore $\{y_n\}$ is a Cauchy sequence, since X is a complete b-metric space, $\exists$ some y in X such that.

$$\lim_{n \rightarrow \infty} Fx_{2n} = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} Gx_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n+2} = y$$

Since S is continuous, therefore $\lim_{n \rightarrow \infty} S^2x_{2n+2} = Sy$ and $\lim_{n \rightarrow \infty} SFx_{2n} = Sy$ and $STx_{2n+1} = Sy$

Since a pair $\{F, S\}$ is compatible, $\lim_{n \rightarrow \infty} d(FSx_{2n}, SFx_{2n}) = 0$ so by lemma (1.5) we have

$$\lim_{n \rightarrow \infty} FSx_{2n} = Sy$$

Putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in (2.1)

$$\begin{aligned}
(2.5) d(FSx_{2n}, Gx_{2n+1}) &\leq \frac{a_1}{k^4} [a_2 \max\{d(S^2x_{2n}, Tx_{2n+1}), d(FSx_{2n}, S^2x_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}(d(S^2x_{2n}, Gx_{2n+1}) + \\
&d(FSx_{2n}, Tx_{2n+1}))\} + (1 - a_2) \{ \max(d^2(S^2x_{2n}, Tx_{2n+1}), d(S^2x_{2n}, FSx_{2n})dGx_{2n+1}, Tx_{2n+1}), \\
&d(S^2x_{2n}, Gx_{2n+1}).d(FSx_{2n}, Tx_{2n+1}), \frac{d(FSx_{2n}, S^2x_{2n})d(FSx_{2n}, Sx_{2n+1})}{2}, \frac{d(Sx_{2n+1}, Gx_{2n+1})d(S^2x_{2n}, Gx_{2n+1})}{2} \}]^{1/2}
\end{aligned}$$

Taking upper limit as $n \rightarrow \infty$ in above and using lemma (1.4)

$$\begin{aligned}
\frac{d(Sy, y)}{k^2} &\leq \lim_{n \rightarrow \infty} \sup d(FSx_{2n}, Gx_{2n+1}) \leq \frac{a_1}{k^4} [a_2 \max\{k^2 d(Sy, y), \frac{k^2}{2}(d(Sy, y) + d(Sy, y))\} + \\
&(1 - a_2) \{ \max(d^2(Sy, y), d(Sy, Sy)d(y, y), d(Sy, y).d(Sy, y), \frac{d(Sy, Sy)d(Sy, y)}{2}, \frac{d(y, y)d(Sy, y)}{2}) \}]^{1/2} \\
&\leq \frac{a_1}{k^4} [a_2 k^2 d(Sy, y) + (1 - a_2) \{ \max(d^2(Sy, y), d^2(Sy, y)) \}]^{1/2} \\
&\leq \frac{a_1}{k^4} [a_2 k^2 d(Sy, y) + (1 - a_2) \{ k^4 (d^2(Sy, y)) \}]^{1/2} \\
&\leq \frac{a_1}{k^4} [a_2 k^2 d(Sy, y) + (1 - a_2) k^2 d(Sy, y)] \\
&\leq \frac{a_1}{k^4} [k^2 d(Sy, y)(a_2 + 1 - a_2)] \\
&\leq \frac{a_1}{k^4} [k^2 d(Sy, y)(a_2 + 1 - a_2)] \\
&\leq \frac{a_1}{k^2} d(Sy, y)
\end{aligned}$$

$$\frac{d(Sy, y)}{k^2} \leq \frac{a_1}{k^2} d(Sy, y)$$

$$d(Sy, y) \leq a_1 d(Sy, y) \text{ and } a_1 \in (0, 1)$$

$$\Rightarrow Sy = y$$

Put $x = x_{2n}$, $y = Tx_{2n+1}$ in (2.1) and using continuity of T

$$\lim_{n \rightarrow \infty} T^2 x_{2n+1} = Ty \text{ and } TGx_{2n+1} = Ty \text{ and Since a pair } \{G, T\} \text{ is compatible then}$$

$$\lim_{n \rightarrow \infty} d(GTx_n, TGx_n) = 0$$

So by lemma (1.5) we have $\lim_{n \rightarrow \infty} GTx_{2n} = Ty$

$$(2.6) d(Fx_{2n}, GTx_{2n+1}) \leq \frac{a_1}{k^4} [a_2 \max\{d(Sx_{2n}, T^2 x_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(GTx_{2n+1}, T^2 x_{2n+1}), \frac{1}{2}(d(Sx_{2n}, GTx_{2n+1}) + d(Fx_{2n}, T^2 x_{2n+1}))\} + (1 - a_2) \{ \max(d^2(Sx_{2n}, T^2 x_{2n+1}), d(Sx_{2n}, Fx_{2n}) d(GTx_{2n+1}, T^2 x_{2n+1}), d(Sx_{2n}, GTx_{2n+1}). d(Fx_{2n}, T^2 x_{2n+1}), \frac{d(Fx_{2n}, Sx_{2n}) d(Fx_{2n}, STx_{2n+1})}{2}, \frac{d(STx_{2n+1}, GTx_{2n+1}) d(Sx_{2n}, GTx_{2n+1})}{2} \}^{\frac{1}{2}}]$$

Taking upper limit as $n \rightarrow \infty$ in above and using lemma (1.4) we obtain

$$\begin{aligned} \frac{d(y, Ty)}{k^2} &\leq \lim_{n \rightarrow \infty} \sup d(Fx_{2n}, GTx_{2n+1}) \leq \frac{a_1}{k^4} [a_2 \max\{k^2 d(y, Ty), \frac{k^2}{2}(d(y, Ty) + d(y, Ty))\} + \\ &(1 - a_2) \{ \max(d^2(y, Ty), d(y, y) d(Ty, Ty), d(y, Ty). d(y, Ty), \frac{d(y, y) d(y, Sy)}{2}, \frac{d(Sy, Ty) d(y, Ty)}{2} \}^{\frac{1}{2}}] \\ &\leq \frac{a_1}{k^4} [a_2 k^2 d(y, Ty) + (1 - a_2) \{ \max(k^4 d^2(y, Ty), d^2(y, Ty), \frac{k^4 d^2 d(y, Ty)}{2} \}^{\frac{1}{2}}] \\ &\leq \frac{a_1}{k^4} [a_2 k^2 d(y, Ty) + (1 - a_2) k^2 d(y, Ty)] \\ &\leq \frac{a_1}{k^4} [(a_2 + 1 - a_2) k^2 d(y, Ty)] \\ &\leq \frac{a_1}{k^2} d(y, Ty) \\ \frac{d(y, Ty)}{k^2} &\leq \frac{a_1}{k^2} d(y, Ty) \end{aligned}$$

$$d(y, Ty) \leq a_1 d(y, Ty) \text{ and } a_1 \in (0, 1)$$

$$\Rightarrow y = Ty$$

Put $x = y$ and $y = x_{2n+1}$ in (2.1) then

$$(2.7) d(Fy, Gx_{2n+1}) \leq \frac{a_1}{k^4} [a_2 \max\{d(Sy, Tx_{2n+1}), d(Fy, Sy), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}(d(Sy, Gx_{2n+1}) + d(Fy, Tx_{2n+1}))\} + (1 - a_2) \{\max(d^2(Sy, Tx_{2n+1}), d(Sy, Fy)d(Gx_{2n+1}, Tx_{2n+1}), d(Sy, Gx_{2n+1}).d(Tx_{2n+1}, Fy), \frac{d(Fy, Sy)d(Fy, Sx_{2n+1})}{2}, \frac{d(Sx_{2n+1}, Gx_{2n+1})d(Sy, Gx_{2n+1})}{2})\}^{\frac{1}{2}}]$$

Taking upper limit as $n \rightarrow \infty$ in above and using lemma (1.9) and using $Sy = Ty = y$ we have

$$\frac{d(Fy, y)}{k^2} \leq \frac{a_1}{k^4} [a_2 \max\{k^2 d(Sy, y), k^2 d(Fy, Sy), k^2 d(y, y), \frac{k^2}{2}(d(Sy, y) + d(Fy, y))\} + (1 - a_2) \{\max(d^2(Sy, y), d(Sy, Fy)d(y, y), d(Sy, y).d(Fy, y), \frac{d(Fy, Sy)d(Fy, y)}{2}, \frac{d(y, y)d(Sy, y)}{2})\}^{\frac{1}{2}}]$$

$$\frac{d(Fy, y)}{k^2} \leq \frac{a_1}{k^4} [a_2 k^2 d(Fy, y) + (1 - a_2) \{\max(k^2 .d(Fy, y), \frac{k^4 d^2(Fy, y)}{2})\}^{\frac{1}{2}}]$$

$$\frac{d(Fy, y)}{k^2} \leq \frac{a_1}{k^4} [a_2 k^2 d(Fy, y) + (1 - a_2) k^2 d(Fy, y)]$$

$$\frac{d(Fy, y)}{k^2} \leq \frac{a_1}{k^4} [d(Fy, y) k^2 (a_2 + 1 - a_2)]$$

$$\frac{d(Fy, y)}{k^2} \leq \frac{a_1}{k^2} d(Fy, y) \quad \text{And } a_1 \in (0, 1)$$

$$d(Fy, y) \leq a_1 d(Fy, y)$$

Which is a contradiction

$$d(Fy, y) = 0$$

$$\Rightarrow y = Fy \quad \text{As } a_1 \in (0, 1)$$

So $Sy = Ty = Fy = y$ we have

$$(2.8) d(y, Gy) = d(Fy, Gy) \leq \frac{a_1}{k^4} [a_2 \max\{d(Sy, Ty), d(Fy, Sy), d(Gy, Ty), \frac{1}{2}(d(Sy, Gy) + d(Fy, Ty))\} + (1 - a_2) \{\max(d^2(Sy, Ty), d(Sy, Fy)d(Gy, Ty), d(Sy, Gy).d(Fy, Ty), \frac{d(Fy, Sy)d(Fy, Sy)}{2}, \frac{d(Sy, Gy)d(Sy, Gy)}{2})\}^{\frac{1}{2}}]$$

$$\leq \frac{a_1}{k^4} [a_2 d(y, Gy) + (1 - a_2) \{\max(d^2(y, y), d(y, y)d(Gy, y), d(y, Gy).d(y, y), \frac{d(y, y)d(y, y)}{2}, \frac{d(y, Gy)d(y, Gy)}{2})\}^{\frac{1}{2}}]$$

$$\leq \frac{a_1}{k^4} [a_2 d(y, Gy) + (1 - a_2) d(y, Gy)]$$

$$\leq \frac{a_1}{k^4} [d(y, Gy)(a_2 + 1 - a_2)]$$

$$d(y, Gy) \leq a_1 d(y, Gy)$$

Which is a contradiction as $a_1 \in (0, 1)$

$$\Rightarrow d(y, Gy) = 0$$

$$\Rightarrow y = Gy$$

Uniqueness of y: if there exist another common fixed point x in X for F, G, S, T

$$d(x, y) = d(Fx, Gy) \leq \frac{a_1}{k^4} [a_2 \max\{d(Sx, Ty), d(Fx, Sx), d(Gy, Ty), \frac{1}{2}(d(Sx, Gy) + d(Fx, Ty))\} + (1 - a_2) \{ \max(d^2(x, y), d(x, x)d(y, y), d(x, y).d(x, y), \frac{d(x, x)d(x, y)}{2}, \frac{d(y, y)d(x, y)}{2}) \}^{\frac{1}{2}}]$$

$$\leq \frac{a_1}{k^4} [a_2 d(x, y) + (1 - a_2) \{ \max(d^2(x, y), d^2(x, y)) \}^{\frac{1}{2}}]$$

$$\leq \frac{a_1}{k^4} [a_2 d(x, y) + (1 - a_2) d(x, y)]$$

$$\leq \frac{a_1}{k^4} d(x, y) < a_1 d(x, y)$$

$$\Rightarrow d(x, y) < a_1 d(x, y)$$

$$\Rightarrow d(x, y) = 0$$

$$\Rightarrow x = y$$

Thus y is a unique common fixed point of F, G, S, T .

Example:

Let $X = [0, \infty)$ be endowed with b metric space (X, d) with metric $d(x, y) = |x - y|^2 = (x - y)^2$, define F, G, S, T on X

$$\text{by } F(x) = \log(1 + \frac{x}{4}), G(x) = \log(1 + \frac{x}{6})$$

$$S(x) = e^{3x} - 1, T(x) = e^{2x} - 1.$$

Obviously $F(x) = G(x) = T(x) = S(x) = [0, \infty)$.

We show that the pair (F, S) is compatible: let $\{x_n\}$ be a sequence in X such that for some $t \in X$

$$\lim_{n \rightarrow \infty} d(Fx_n, t) = 0 \text{ and } t \in X \lim_{n \rightarrow \infty} d(Sx_n, t) = 0$$

i.e $\lim_{n \rightarrow \infty} |Fx_n - t| = 0$, $\lim_{n \rightarrow \infty} |Sx_n - t| = 0$. since F and S are continuous, we have

$$\lim_{n \rightarrow \infty} d(FSx_n, SFx_n) = \lim_{n \rightarrow \infty} |FSx_n - SFx_n|^2 = |Ft - St|^2 = \left| \log\left(1 + \frac{t}{4}\right) - (e^{3t} - 1) \right|^2 = 0 \Leftrightarrow t = 0$$

Then (F, S) are compatible.

Similarly {G, T} are compatible.

For each $x, y \in X$

$$\begin{aligned} d(Fx, Gy) &= (Fx - Gy)^2 = \left[\log\left(1 + \frac{x}{4}\right) - \log\left(1 + \frac{y}{6}\right) \right]^2 \leq \left(\frac{x}{4} - \frac{y}{6} \right)^2 \leq \frac{1}{(12)^2} [(3x - 2y)^2] \\ &\leq \frac{1}{(12)^2} [e^{3x} - e^{2y}]^2 \\ &= \frac{1}{144} [d(Sx, Ty)] \\ &\leq \frac{1}{2^4} [a_2 \max\{d(Sx, Ty), d(Sx, Sx), d(Ty, Ty), \frac{1}{2}(d(Sx, Ty) + d(Sx, Ty))\} + (1 - a_2) \{ \max(d^2(Sx, Ty), \\ &d(Sx, Sx)d(Ty, Ty), d(Sx, Ty).d(Sx, Ty), \frac{d(Sx, Sx)d(Sx, Sy)}{2}, \frac{d(Sy, Ty)d(Sx, Ty)}{2} \})^{\frac{1}{2}}] \end{aligned}$$

$$\text{Where } a_1 = \frac{1}{9} \in (0, 1) \text{ and } k=2$$

Thus F, G, S, T satisfy all condition of theorem (2.1). moreover 0 is the unique common fixed point of F, G, S, T.

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