Some Fixed Point Theorems in $\varphi - \psi$ weak contraction on Fuzzy Metric Space

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Abstract—In this paper, we discuss some results on fixed point theorems in $\varphi - \psi$ weak contraction on fuzzy metric spaces, which are study of generalisation of some existing results are also given in the form of corollary.

Keywords—fuzzy metric space, continuous t-norm, $\varphi - \psi$ weak contraction

I. INTRODUCTION


$$\varphi(M(Tx, Ty, t)) \geq k(t) \cdot \varphi(M(x, y, t)), \forall x, y \in X, t > 0,(1.1)$$

obtained fixed point result for self-mapping of T. Recently, many authors using altering distance function and give their contribution in various metric spaces [3, 4].

In this paper, we proved some fixed point theorems in $\varphi - \psi$ weak contraction on fuzzy metric spaces, which are our study of generalisation of some existing results.

Definition 1.1 A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = (x, \mu_A(x))$; $x \in A, \mu_A(x) \in [0,1]$. In the pair $(x, \mu_A(x))$, the first element $x$ belongs to the classical set $A$, the second element $\mu_A(x)$ belongs to the interval $[0,1]$, is called the membership function.

Definition 1.2 A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions:
1. * is associative and commutative,
2. * is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$.
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$

Example 1.3
1. Lukasievicz t-norm: $a * b = \max(a + b - 1,0)$
2. Product t-norm: $a * b = a \cdot b$
3. Minimum t-norm: $a * b = \min(a, b)$

Definition 1.4 A fuzzy metric space is an ordered triple $(X, M, *)$ such that $X$ is a nonempty set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $X \times X \times (0, \infty) \rightarrow [0,1]$ satisfies the following conditions: $\forall x, y, z \in X$ and $s, t > 0$
1. $M(x, y, 0) = 0, t > 0$
2. $M(x, y, t) = 1$ if and only if $x = y, t > 0$
3. $M(x, y, t) = M(y, x, t)$

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4. \( M(x, z, t + s) \geq M(x, y, t) \ast M(y, z, s) \)
5. \( M(x, y, \cdot) : [0, \infty) \to [0,1] \) is left-continuous.

Then \( M \) is called a fuzzy metric on \( X \).

**Definition 1.5** A fuzzy metric space is an ordered triple such that \( X \) is a non-empty set, \( \ast \) is a continuous \( t \)-norm and \( M \) is a fuzzy set on \( X \times X \times (0, \infty) \to [0,1] \) satisfies the following conditions:

1. \( M(x, y, t) > 0 \), \( t > 0 \)
2. \( M(x, y, t) = 1 \) if and only if \( x = y \) and \( t > 0 \)
3. \( M(x, y, t) = M(y, x, t) \)
4. \( M(x, z, t + s) \ast M(y, z, s) \leq M(x, z, t + s) \)
5. \( M(x, y, \cdot) : (0, \infty) \to [0,1] \) is continuous.

Then \( M \) is called a fuzzy metric on \( X \).

**Definition 1.6** Let \( (X, M, \ast) \) be a fuzzy metric space, for \( t > 0 \) the open ball \( B(x, r, t) \) with a centre \( x \in X \) and a radius \( 0 < r < 1 \) is defined by

\[
B(x, r, t) = y \in X : M(x, y, t) > 1 - r.
\]

A subset \( A \subset X \) is called open if for each \( x \in A \), there exist \( t > 0 \) and \( 0 < r < 1 \) such that \( B(x, r, t) \subset A \). Let \( \tau \) denote the family of all open subsets of \( X \). Then \( \tau \) is topology on \( X \), called the topology induced by the fuzzy metric \( M \).

**Definition 1.7** Let \( (X, M, \ast) \) be a fuzzy metric space

1. A sequence \( x_n \) in \( X \) is said to be convergent to a point \( x \) in \( (X, M, \ast) \) if \( \lim_{t \to \infty} M(x, y, t) = 1 \) for all \( t > 0 \).
2. sequence \( x_n \) in \( X \) is called a Cauchy sequence in \( (X, M, \ast) \), if for each \( 0 < \varepsilon < 1 \) and \( t > 0 \), there exists \( n_0 \in N \) such that \( M(x_n, x_m, t) > 1 - \varepsilon \) for each \( n, m \geq n_0 \).
3. A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
4. A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

**Lemma 1.8** Let \( (X, M, \ast) \) be a fuzzy metric space. For all \( u, v \in X, M(u, v, \cdot) \) is non-decreasing function.

**Proof.** If \( M(u, v, t) > M(u, v, s) \) for some \( 0 < t < s \).

Then \( M(u, v, t) \ast M(v, s - t) \leq M(u, v, s) < M(u, v, t) \).

Thus \( M(u, v, t) < M(u, v, t) < M(u, v, t) \),

(since \( M(v, v, s - t) = 1 \))

which is a contradiction.

**Definition 1.9** A function \( \varphi : [0,1] \to [0,1] \) is called a control function or an altering distance function if it satisfies the following properties:

(CF1). \( \varphi \) is strictly decreasing and continuous;

(CF2). \( \varphi(\lambda) \geq 0 \), \( \forall \lambda \neq 1 \) if \( \varphi(\lambda) = 0 \) if and only if \( \lambda = 1 \). It is obvious that \( \lim_{\lambda \to 1^-} \varphi(\lambda) = \varphi(1) = 0 \).

where \( \varphi \) in class of function \( \Phi \).

**II. MAIN RESULTS**

**Theorem 2.1** Let \( (X, M, \ast) \) be a complete strong fuzzy metric space with continuous \( t \)-norm \( \ast \) and let \( T \) is a self-mapping in \( X \) such that

\[
\varphi(M(Tu, Tv, t)) \leq \varphi(M(u, Tu, t)) + M(v, Tv, t) + M(Tu, v, t)
\]

\[
+ M(u, v, t) + M(v, v, t) + \max(M(u, Tu, t), M(v, Tv, t)) - \psi(M(u, Tu, t) + M(v, Tv, t) + M(Tu, v, t) +
\]

\[
M(u, Tu, t) + M(u, v, t) + \max(M(u, Tu, t), M(v, Tv, t)),
\]

where \( \varphi \) and \( \psi \) are altering distance function and ultra altering distance function respectively, \( \varphi(t + s) \leq \varphi(t) + \varphi(s) \) for all \( t, s \in S \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** Let \( u \) be any arbitrary point in \( X \) and define a sequence \( u_n \in X \) such that \( u_{n+1} = Tu_n \).

Assume that \( u_{n+1} = Tu_n = u_n \) for some \( n \in \mathbb{N} \), then \( u_n \) is a fixed point of \( T \).
Suppose \( u_{n+1} \neq u_n \), put \( u = u_{n-1} \) and \( v = u_n \) in equation (2.1) we get

\[
\varphi(M(Tu_{n-1}, Tu_{n}, t)) \leq \varphi(M(u_{n-1}, Tu_{n-1}, t) + M(u_n, Tu_n, t) + M(Tu_{n-1}, Tu_n, t)(M(u_{n-1}, Tu_{n-1}, t)
\]
\[
+\max(M(u_{n-1}, Tu_{n-1}, t), M(u_n, Tu_n, t)))
\]
\[
-\varphi(M(u_{n-1}, Tu_{n-1}, t) + M(u_n, Tu_n, t))
\]
\[
+M(Tu_{n-1}, Tu_n, t)M(u_{n-1}, Tu_{n-1}, t) + M(u_n, Tu_n, t)
\]
\[
+\max(M(u_{n-1}, Tu_{n-1}, t), M(u_n, Tu_n, t))
\]
\[
(2.2)
\]

\[
\leq \varphi(M(u_{n-1}, u_n, t)) + (M(u_{n-1}, u_n, t)) + (M(u_n, u_n, t)) + (M(u_{n-1}, u_n, t)) + (M(u_n, u_n, t))
\]
\[
+\max(M(u_{n-1}, u_n, t), M(u_n, u_n, t)) - \varphi(M(u_{n-1}, u_n, t)) + (M(u_n, u_n, t)) + (M(u_n, u_n, t))
\]
\[
+(M(u_{n-1}, u_n, t)) + (M(u_n, u_n, t)) + \max(M(u_{n-1}, u_n, t), M(u_n, u_n, t))
\]
\[
(2.3)
\]

Using above inequalities in (2.3) we get,

\[
\varphi(M(u_{n-1}, u_{n+1}, t)) \geq \varphi((M(u_{n-1}, u_n, t)) + (M(u_n, u_{n+1}, t)))
\]
\[
(2.4)
\]

If \( \max(M(u_{n-1}, u_n, t), M(u_n, u_{n+1}, t)) = M(u_{n-1}, u_n, t) \).

Then the above inequality (2.4) becomes

Continuing this process, we get,

\[
\varphi(M(u_n, u_{n+1}, t)) \leq \varphi(M(u_{n-1}, u_{n+1}, t)) \leq \varphi(M(u_{n-1}, u_n, t)) < \varphi(M(u_n, u_{n+1}, t))
\]
\[
(2.7)
\]

Similarly,

\[
\text{If } \max(M(u_{n-1}, u_n, t), M(u_n, u_{n+1}, t)) = M(u_n, u_{n+1}, t)
\]

Then the inequality (2.5) becomes

\[
\varphi(M(u_n, u_{n+1}, t)) \leq \varphi(M(u_{n-1}, u_n, t)) \leq \varphi(M(u_n, u_{n+1}, t))
\]
\[
(2.8)
\]

Hence \( \varphi(M(u_n, u_{n+1}, t)) \leq \varphi(M(u_{n-1}, u_n, t)) < \varphi(M(u_{n-1}, u_n, t)) \).

This gives \( M(u_n, u_{n+1}, t) > M(u_{n-1}, u_n, t) \).
Since the sequence \( M(u_n, u_{n+1}, t) \) is non decreasing
Taking limit \( n \to \infty \), we get

\[
\lim_{n \to \infty} M(u_n, u_{n+1}, t) = q(r), \text{ for } q: (0, \infty) \to [0,1] \tag{2.10}
\]

Suppose that \( q(r) \neq 1 \) for some \( r > 0 \) as \( n \to \infty \),
Now (2.7) becomes,

\[
\varphi(q(r)) \leq q(q(r)) < \varphi(q(r)) \tag{2.11}
\]

which is a contradiction.
Hence

\[
\lim_{n \to \infty} M(u_n, u_{n+1}, t) = 1, t > 0
\]

Next we prove that the sequence \( u_n \) is a Cauchy’s sequence.
Assume that \( u_n \) is not a Cauchy’s sequence then for any \( 0 < \varepsilon < 1, t > 0 \) then there exists sequence \( u_{n_k} \) and \( u_{m_k} \) where \( n_k, m_k \geq n \) and \( n_k, m_k \in \mathbb{N} \) such that

\[
M(u_{n_k}, u_{m_k}, t) \leq 1 - \varepsilon \tag{2.12}
\]

Let \( n_k \) be least integer exceeding \( m_k \) satisfying the above property

\[
M(u_{n_k-1}, u_{m_k}, t) > 1 - \varepsilon, \quad n_k, m_k \in \mathbb{N} \quad \text{ and } \quad t > 0 \tag{2.13}
\]

Put \( u = u_{n_k-1} \) and \( v = u_{m_k-1} \)

\[
\varphi(M(Tu_{n_k-1}, Tu_{m_k-1}, t)) \leq \varphi((M(u_{n_k-1}, Tu_{n_k-1}, t)) + (M(u_{m_k-1}, Tu_{m_k-1}, t)) + (M(Tu_{n_k-1}, u_{m_k-1}, t))(M(u_{n_k-1}, Tu_{m_k-1}, t)) + (M(u_{n_k-1}, u_{m_k-1}, t)) + \max(M(u_{n_k-1}, Tu_{n_k-1}, t), M(u_{m_k-1}, Tu_{m_k-1}, t))) \tag{2.14}
\]

\[
-\psi((M(u_{n_k-1}, u_{n_k-1}, t))(M(u_{m_k-1}, u_{m_k-1}, t)) + (M(u_{n_k-1}, u_{m_k-1}, t))(M(u_{m_k-1}, u_{n_k-1}, t)) + (M(u_{n_k-1}, u_{n_k}, t)) + \max(M(u_{n_k-1}, Tu_{n_k-1}, t), M(u_{m_k-1}, Tu_{m_k-1}, t)))
\]

\[
\varphi(M(u_{n_k-1}, u_{m_k-1}, t)) \leq \varphi((M(u_{n_k-1}, u_{n_k-1}, t)) + (M(u_{m_k-1}, u_{m_k-1}, t)) + (M(u_{n_k-1}, u_{m_k-1}, t)) + \max(M(u_{n_k-1}, u_{n_k}, t), M(u_{m_k-1}, u_{m_k}, t)))
\]

\[
-\psi((M(u_{n_k-1}, u_{n_k}, t)) + (M(u_{m_k-1}, u_{m_k}, t))) + (M(u_{n_k}, u_{m_k}, t))(M(u_{n_k-1}, u_{m_k-1}, t)) + (M(u_{n_k}, u_{m_k}, t))(M(u_{n_k-1}, u_{m_k}, t)) + \max(M(u_{n_k-1}, u_{n_k}, t), M(u_{m_k-1}, u_{m_k}, t)))
\]

If \( \max(M(u_{n_k-1}, u_{n_k}, t), M(u_{m_k-1}, u_{m_k}, t)) = M(u_{n_k-1}, u_{n_k}, t) \)
\[
\varphi(M(u_{nk}, u_{mk}, t)) \leq \varphi((M(u_{nk-1}, u_{nk}, t)) + (M(u_{mk-1}, u_{mk}, t))
+ (M(u_{nk-1}, u_{nk}, t))(M(u_{nk-1}, u_{mk}, t)) + 
+ (M(u_{nk-1}, u_{mk-1}, t)) + (M(u_{nk-1}, u_{nk}, t)))
- \psi((M(u_{nk-1}, u_{nk}, t)) + (M(u_{mk-1}, u_{mk}, t))
(M(u_{nk}, u_{mk-1}, t))(M(u_{nk-1}, u_{mk}, t))
+ (M(u_{nk-1}, u_{nk-1}, t)) + (M(u_{nk-1}, u_{nk}, t))) 
\leq \varphi((M(u_{nk-1}, u_{nk}, t)) + (M(u_{mk-1}, u_{mk}, t))
+ (M(u_{nk-1}, u_{nk}, t))(M(u_{nk-1}, u_{mk}, t))
+ (M(u_{nk-1}, u_{mk-1}, t)) + (M(u_{nk-1}, u_{nk}, t))) 
(2.15)
\]

Also (2.13) and (CF1) we get

\[
\varphi(M(u_{nk-1}, u_{mk-1}, t)) \leq \varphi(M(u_{nk-1}, u_{nk}, t)) + \varphi(M(u_{nk-1}, u_{mk-1}, t)) 
\leq \varphi(M(u_{nk-1}, u_{nk}, t)) + \varphi(M(u_{nk-1}, u_{mk}, t)) + 
\varphi(M(u_{nk, u_{mk-1}, t})) \varphi(M(u_{nk-1, u_{mk}}))
+ \varphi(M(u_{nk-1, u_{mk}, t})) + \varphi(M(u_{nk-1, u_{nk}})) 
+ \varphi(M(u_{nk, u_{mk-1}, t})) + \varphi(M(u_{nk-1, u_{nk}})) 
\leq \varphi(M(u_{nk-1, u_{mk}})) \leq \varphi(1 - \varepsilon). 
(2.16)
\]

Applying the previous inequalities we get

\[
\varphi(M(u_{nk-1, u_{mk}}, t)) \leq \varphi(M(u_{nk-1}, u_{nk}, t)) + \varphi(M(u_{nk-1}, u_{mk}, t)) + 
\varphi(M(u_{nk, u_{mk-1}, t})) \varphi(M(u_{nk-1, u_{mk}}))
+ \varphi(M(u_{nk-1, u_{mk}, t})) + \varphi(M(u_{nk-1, u_{nk}})) 
+ \varphi(M(u_{nk, u_{mk-1}, t})) + \varphi(M(u_{nk-1, u_{nk}})) 
\leq \varphi(M(u_{nk-1, u_{mk}})) \leq \varphi(1 - \varepsilon). 
(2.17)
\]

Substituting (2.16), (2.17), and (2.18) in (2.15) we have

\[
\varphi(M(u_{nk}, u_{mk}, t)) \leq \varphi(M(u_{nk-1}, u_{nk}, t)) + \varphi(M(u_{mk-1}, u_{mk}, t))
+ (M(u_{nk-1}, u_{nk}, t))(M(u_{nk-1}, u_{mk}, t)) + (M(u_{nk-1}, u_{mk-1}, t)) + (M(u_{nk-1}, u_{nk}, t))) 
- \psi(M(u_{nk-1, u_{nk}, t}) + (M(u_{mk-1}, u_{mk}, t))
(M(u_{nk}, u_{mk-1}, t))(M(u_{nk-1}, u_{mk}, t))
+ (M(u_{nk-1}, u_{nk-1}, t)) + (M(u_{nk-1}, u_{nk}, t))) 
\leq \varphi((M(u_{nk-1}, u_{nk}, t)) + (M(u_{mk-1}, u_{mk}, t))
+ (M(u_{nk-1}, u_{nk}, t))(M(u_{nk-1}, u_{mk}, t))
+ (M(u_{nk-1}, u_{mk-1}, t)) + (M(u_{nk-1}, u_{nk}, t))) 
(2.19)
\]

Using (2.12) we obtain,

\[
\varphi(M(u_{nk}, u_{mk}, t)) \leq \varphi(1 - \varepsilon) 
(2.20)
\]

\[
\varphi(1 - \varepsilon) \leq \varphi(M(u_{nk-1}, u_{nk}, t))
+ (M(u_{mk-1}, u_{mk-1}, t)) \varphi(1 - \varepsilon) 
(2.21)
\]

Taking \( k \to \infty \) in above inequality we obtain

\[
\varphi(1 - \varepsilon) \leq \varphi(1 - \varepsilon) 
(2.22)
\]
Which is a contradiction, \( \varepsilon > 0 \)
Hence \( u_n \) is a Cauchy’s sequence.
Since \( X \) is complete and there exist \( z \in X \) such that \( \lim_{n \to \infty} u_n = z \)
That is \( M(u_n, z, t) = 1 \) as \( n \to \infty \)
Put \( u = u_{n-1} \) and \( v = z \) in equation (2.1) we get
\[
\varphi(M(u_n, Tz, t)) \leq \varphi(M(u_{n-1}, u, t) + (M(z, Tz, t)) + (M(u_{n-1}, Tz, t)) + \max(M(u_{n-1}, u, t), M(z, Tz, t)))
\]  
(2.23)

Taking \( n \to \infty \) in (2.23)
\[
\varphi(M(z, Tz, t)) \leq 0, \quad t > 0
\]  
(2.24)

Therefore, \( M(z, Tz, t) = 1 \), and \( z = Tz \).
To prove Uniqueness,
Suppose that \( w \) is another fixed point of \( T \), that is \( Tw = w \) where \( q \neq z \)
\[
\varphi(M(z, w, t)) \leq 0, \quad t > 0
\]  
(2.25)

Hence \( z = w \) is the unique fixed point of \( T \).

**Corollary 2.2** Let \((X, M, \ast)\) be a complete strong fuzzy metric space with continuous \( t \)-norm \( \ast \) and let \( T \) is a self-mapping in \( X \).
If there exists a control function \( \varphi \) and \( \theta(t) \), such that
\[
\varphi(M(Tu, Tv, t)) \leq \varphi(M(u, Tu, t)) + (M(v, Tv, t)) + (M(Tu, v, t)) + (M(u, v, t)) \]
\[
-\psi(M(u, Tu, t)) + (M(v, Tv, t)) + (M(Tu, v, t)) + (M(u, v, t))
\]  
(2.26)

Then \( T \) has a unique fixed point in \( X \).

**Proof.** The proof of the above theorem (2.1) considering the fuzzy contraction on the fuzzy metric space \((X, M, \ast)\),
\[
\varphi(M(Tu, Tv, t)) \leq \varphi(M(u, Tu, t)) + (M(v, Tv, t)) + (M(Tu, v, t)) + (M(u, v, t)) \]
\[
-\psi(M(u, Tu, t)) + (M(v, Tv, t)) + (M(Tu, v, t)) + (M(u, v, t)).
\]  

### III. Reference


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