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Research Article

Length-Biased Distribution for Family of Life-Time Testing Models and its Properties

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Abstract— Our research introduces a new lifetime distribution family called Length-Biased Lifetime Distribution, which is based on length bias. To comprehend the nature of the suggested distribution, we take into account its many statistical aspects, such as its reliability, hazard function and other characteristics. Also, to estimate the parameters of the suggested distribution, we have utilized the ML technique. The Lorenz and Bonferroni curves, Shannon's entropy, and Renyi entropy are also obtained. Using two real-life datasets, we evaluate the suggested distribution's performance to that of competing distributions.

Keywords— Entropy, Hazard Function, Length-Biased Distribution, Maximum Likelihood Estimation, Reliability, Reversed Hazard Function, Order Statistic.

1. Introduction

Weighted distributions were originally introduced by Fisher (1934) [1], who studied the potential effects of ascertainment procedures on the form of the distribution of observed data. These distributions are used in various studied pertaining to branching processes, reliability, ecology, and biomedicine. Rao (1965) [2] introduced more generalized approach for modeling the statistical data, which doesn't solely relying on standard distributions. When the weighted function specifically accounts for the length of units, this weighted distribution simplifies to what we commonly refer to as the length-biased distribution. This concept was first coined by Cox (1969) [3] and Zelen (1974) [4]. "Size-biased" refers to a distribution in which the sampling process selects units with a probability proportional to a certain measure of their size. The complete description of weighted distributions can be found in many reliable sources. In the literature, there are numerous recently discovered distributions and their weighted counterparts, whose statistical behaviour has been thoroughly investigated for decades. Shaban and Boudrissa (2007) [5], Priyadarshani (2011) [6] have obtained the size-biased of Weibull distribution and studied its properties in details. Das and Roy (2011) [7] presented the idea of length-biased sampling and weighted distributions by highlighting some of the circumstances in which the underlying models maintain their form. They also developed the length-biased version of the weighted Weibull distribution and discussed the length-biased Weighted Generalized Rayleigh distribution and its properties. The statistical characteristics of the weighted exponential distribution and its length-biased variant were examined by Das and Kundu (2016) [8]. The weighted transmuted power distribution was derived by Dar et al. (2018) [9], who also presented its features and uses. The length-biased Sushila distribution with a range of statistical characteristics and its uses were covered by Rather and Subramanian (2018) [10]. For a concise review of the literature on size-biased and length-biased distributions, one may refer, Rather and Subramanian (2018) [11], Malik and Ahmad (2018) [12], Atikankul *et al.* (2020) [13], Chaito *et al.* (2022) [14], Al-Omari *et al.* (2023) [15] and Hassan & Muhammed (2024) [16].

A r.v. T, is said to have a length biased distribution if its pdf is defined as

$$f_l(t) = \frac{w(t)\, f(t)}{\mu} \tag{1.1}$$

where, $\mu = \int w(t)\, f(t) dt$, provided that $w(t) = t$.
The *pdf* and *cdf* of the life time distribution is

$$f(t) = \frac{\xi t^{\omega\xi-1}}{\zeta^{\omega\eta}\Gamma(\omega)} exp\left(\frac{t^{\xi}}{\zeta^{\eta}}\right); \qquad t, \omega, \eta, \xi, \zeta > 0 \tag{1.2}$$

where ζ is the scale parameter and ω, η, ξ are shape parameters

$$F(t) = \frac{\gamma\left(\omega, \frac{t^{\xi}}{\zeta^{\eta}}\right)}{\Gamma(\omega)}; \qquad t, \omega, \eta, \xi, \zeta > 0 \tag{1.3}$$

where, $\Gamma(\omega) = \int_x^\infty t^{\omega-1}e^{-t}dt$ and $\gamma(y,\ x) = \int_0^x t^{y-1}e^{-t}dt$ the complete and lower incomplete gamma functions, respectively.

In the present study a new distribution proposed known as length-biased lifetime (LBLT) distribution. The paper organized as: Section 2 is the brief introduction of the distribution and its related characteristics. The text discusses the use of the maximum likelihood technique in Section 3 for the purpose of estimating the parameters of the model. Section 4 discusses Entropy, Bonferroni and Lorenz curves while Section 5 focuses on the practical uses of two actual data sets. Ultimately, the conclusion is clearly laid out in section 6.

2. Proposed Length-biased Lifetime distribution

This section deals with the formation of the Length-Biased Lifetime (LBL) distribution. In this suggested new distribution, we provide the pdf, cdf and analyse a number of characteristics. On using the equations (1.1) and (1.2), we obtained the following density of LBLT distribution

$$f_l(t) = \frac{\xi t^{\omega\xi} \exp\left(-\frac{t^\xi}{\zeta\eta}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}; \qquad t,\omega,\eta,\xi,\zeta > 0 \qquad (2.1)$$

The various density shapes of the proposed LBLT distribution are obtained by taking different values of ω, η, ξ and ζ, are shown in Fig. (1). Also, it is clearly shows that the distribution is positively skewed.

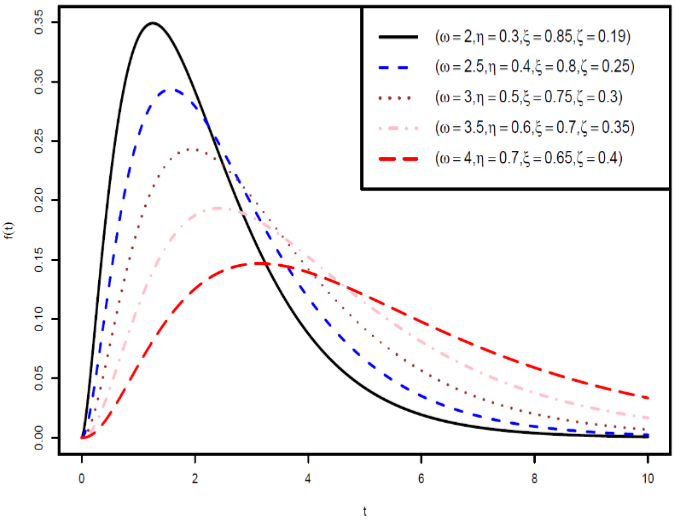

Figure 1: pdf plots for LBLT distribution

It is interested to mention that the distributions listed below are specific cases of the LBLT distribution:

- For ω = η = ξ = 1 it follows Length-biased one parameter Exponential distribution.
- For η = ξ = 1 it follows Length-biased Gamma distribution.
- For η = ξ it follows Length-biased generalized Gamma distribution.
- For η = ξ = 1 and ω is positive integer it follows Length-biased Erlang distribution.
- For ω = 1, η = ξ it follows Length-biased Weibull distribution.
- For ω = 1/2, η = ξ = 2 it follows Length-biased half Normal distribution.
- For ω = η = 1, ξ = 2 it follows Length-biased Chi distribution.
- For ω = ω/2, η = 1, ξ = 2 it follows Length-biased Rayleigh distribution.
- For ω = 3/2, η = 1, ξ = 2 it follows Length-biased Maxwell's failure distribution.

The *cdf* of Length-biased Lifetime (LBLT) distribution is given by

$$F_l(t) = \frac{\gamma\left(\omega+\frac{1}{\xi},\ \frac{t^\xi}{\zeta\eta}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}; \qquad t,\omega,\eta,\xi,\zeta > 0 \qquad (2.2)$$

2.1. Reliability Characteristics

2.1.1. Reliability Function

The Reliability function R(t) of Length-biased Lifetime (LBLT) distribution is given by

$$R(t) = \frac{\Gamma\left(\omega+\frac{1}{\xi},\ \frac{t^\xi}{\zeta\eta}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}; \qquad t,\omega,\eta,\xi,\zeta > 0 \qquad (2.3)$$

where, $\Gamma(y,\ x) = \int_x^\infty t^{y-1}e^{-t}dt.$

The different shapes of reliability are obtained by taking different values of parameters are shown in Fig. (2).

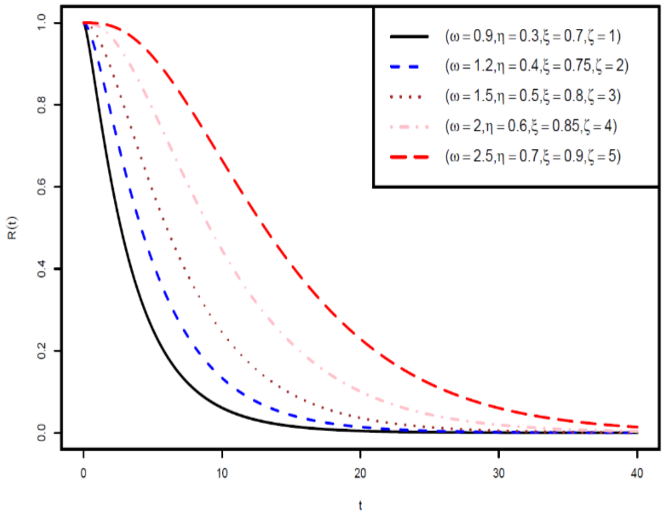

Figure 2: reliability plots for LBLT distribution

It is observed that from Fig. (2) and Table (1) for different values of different parameters the reliability decreases when the value of t increases.

Table 1: Reliability tables

For fixed $\eta = 3,\ \xi = 2,\ \zeta = 1.5$			
t	ω = 1	ω = 2	ω = 3
1	0.89813	0.98834	0.99903
2	0.49917	0.79588	0.93654
3	0.14895	0.37657	0.61936
4	0.02353	0.09133	0.21991
5	0.00198	0.01118	0.03845
For fixed ω = 3, ξ = 2, ζ = 1.5			
t	η = 1	η = 2	η = 3
1	0.98753	0.99643	0.99903
2	0.61936	0.82931	0.93654
3	0.10056	0.33259	0.61936
4	0.00331	0.04737	0.21991
5	2.295E-05	0.00233	0.03845
For fixed ω = 3, η = 2, ζ = 1.5			
t	ξ = 1	ξ = 2	ξ = 3
1	0.99483	0.99945	0.99995
2	0.77978	0.95984	0.99483
3	0.25266	0.72072	0.94472
4	0.02512	0.33259	0.77978
5	0.00076	0.08527	0.51088
For fixed ω = 3, η = 2, ξ = 1			
t	ζ = 1	ζ = 2	ζ = 3
1	0.98101	0.99987	0.99999
2	0.85712	0.99825	0.99992
3	0.64723	0.99271	0.99961
4	0.43347	0.98102	0.99886
5	0.26503	0.96173	0.99744

2.1.2. Hazard Function

The hazard function is described as

$$h(t) = \frac{f(t)}{R(t)}$$

$$h(t) = \frac{\dfrac{\xi t^{\omega\xi} \exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)} \Gamma\left(\omega+\frac{1}{\xi}\right)}}{\dfrac{\Gamma\left(\omega+\frac{1}{\xi}, \frac{t^{\xi}}{\zeta^{\eta}}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}}$$

On simplifying, we get

$$h(t) = \frac{\xi t^{\omega\xi} \exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)} \Gamma\left(\omega+\frac{1}{\xi}, \frac{t^{\xi}}{\zeta^{\eta}}\right)} \tag{2.4}$$

Different shapes of hazard function are obtained by taking different values of parameters. From Fig. (3), it's clearly shows that hazard rate is upside-down for the LBLT distribution.

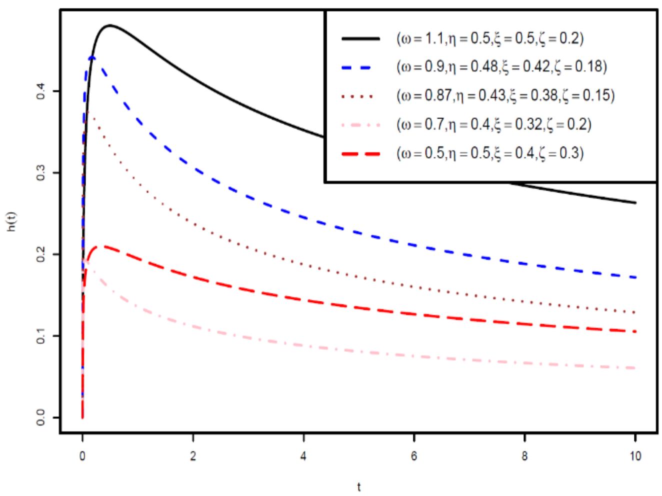

Figure 3: Hazard plots for LBLT distribution

2.1.3. Reversed Hazard Function

The reverse hazard rate is defined as

$$R_h(t) = \frac{f(t)}{F(t)}$$

$$R_h(t) = \frac{\dfrac{\xi t^{\omega\xi} exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)} \Gamma\left(\omega+\frac{1}{\xi}\right)}}{\dfrac{\gamma\left(\omega+\frac{1}{\xi}, \frac{t^{\xi}}{\zeta^{\eta}}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}}$$

On simplifying, we get

$$R_h(t) = \frac{\xi t^{\omega\xi} exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)} \gamma\left(\omega+\frac{1}{\xi}, \frac{t^{\xi}}{\zeta^{\eta}}\right)} \tag{2.5}$$

2.2. Statistical Properties

Theorem 2.1. For $r = 1, 2, \ldots, r^{th}$ moment of random variable T is given by

$$\mu'_r = \zeta^{\eta r/\xi} \frac{\Gamma\left(\omega+\frac{r+1}{\xi}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)} \tag{2.6}$$

Proof. On using the pdf of the r.v. T given at equation (2.1), we get

$$\mu'_r = \int_0^{\infty} t^r f(t) dt$$

$$= \int_0^{\infty} t^r \frac{\xi t^{\omega\xi} exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)} \Gamma\left(\omega+\frac{1}{\xi}\right)} dt \tag{2.7}$$

theorem follows by using transformation $y = \frac{t^{\xi}}{\zeta^{\eta}}$, and basic concept of gamma function in equation (2.7).

Lemma 2.1. If a r.v. T follows LBLT distribution then on substituting $r = 1, 2$ in equation (2.6), we obtain the mean and variance, respectively.

$$mean = \mu'_1 = \zeta^{\eta/\xi} \frac{\Gamma\left(\omega+\frac{2}{\xi}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}$$

$$\mu'_2 = \zeta^{2\eta/\xi}\frac{\Gamma\left(\omega+\frac{3}{\xi}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}$$

And

$Variance =$

$$\mu_2 = \frac{\zeta^{2\eta/\xi}}{\left(\Gamma\left(\omega+\frac{1}{\xi}\right)\right)^2}\left[\Gamma\left(\omega+\frac{3}{\xi}\right)\Gamma\left(\omega+\frac{1}{\xi}\right)-\left(\Gamma\left(\omega+\frac{2}{\xi}\right)\right)^2\right]$$

Lemma 2.2. If a r.v. T follows LBLT distribution then the C.V. is given by

$$\frac{\left[\Gamma\left(\omega+\frac{3}{\xi}\right)\Gamma\left(\omega+\frac{1}{\xi}\right)-\left(\Gamma\left(\omega+\frac{2}{\xi}\right)\right)^2\right]}{\Gamma\left(\omega+\frac{2}{\xi}\right)} \tag{2.8}$$

Proof. Coefficient of variation is given by,

$$C.V. = \frac{\sqrt{\mu_2}}{\mu'_1}$$

$$= \frac{\sqrt{\frac{\zeta^{2\eta/\xi}}{\left(\Gamma\left(\omega+\frac{1}{\xi}\right)\right)^2}\left[\Gamma\left(\omega+\frac{3}{\xi}\right)\Gamma\left(\omega+\frac{1}{\xi}\right)-\left(\Gamma\left(\omega+\frac{2}{\xi}\right)\right)^2\right]}}{\zeta^{\eta/\xi}\frac{\Gamma\left(\omega+\frac{2}{\xi}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}}$$

$$= \frac{\left[\Gamma\left(\omega+\frac{3}{\xi}\right)\Gamma\left(\omega+\frac{1}{\xi}\right)-\left(\Gamma\left(\omega+\frac{2}{\xi}\right)\right)^2\right]}{\Gamma\left(\omega+\frac{2}{\xi}\right)}$$

Lemma 2.3. If a random variable T follows LBLT distribution then characteristic function (*cf*) and moment generating function (*mgf*) of T are respectively, given as

$$\phi_x(t) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\zeta^{\eta r/\xi}\frac{\Gamma\left(\omega+\frac{r+1}{\xi}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)} \tag{2.9}$$

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\zeta^{\eta r/\xi}\frac{\Gamma\left(\omega+\frac{r+1}{\xi}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)} \tag{2.10}$$

On using equation (2.1) and Taylor's series expansion the Lemma 2.3, follows.

Theorem 2.2. If T ~ LBLT(ω,η,ξ,ζ), then the harmonic mean of T is given by

$$\frac{1}{H} = \frac{\Gamma(\omega)}{\zeta^{\eta}\Gamma\left(\omega+\frac{1}{\xi}\right)} \tag{2.11}$$

Proof. The harmonic mean (H) is described as

$$\frac{1}{H} = E\left(\frac{1}{T}\right)$$

$$= \int_0^{\infty}\frac{1}{t}f(t)dt$$

using equation (2.1), we get

$$= \int_0^{\infty}\frac{1}{t}\frac{\xi t^{\omega\xi}\exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}dt$$

$$= \int_0^{\infty}\frac{\xi t^{\omega\xi-1}\exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}dt \tag{2.12}$$

theorem, follows on using the transformation $y = \frac{t^{\xi}}{\zeta\eta}$ in equation (2.12).

Theorem 2.3. If T ~ LBLT(ω,η,ξ,ζ), then the mode of T is given by

$$t^* = (\omega\zeta^{\eta})^{1/\xi} \tag{2.13}$$

Proof. We get the mode by solving $\frac{d}{dt}f(t) = 0$, by differentiate equation (2.1), we get

$$\frac{d}{dt}f(t) = \frac{\xi}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}\frac{d}{dt}t^{\omega\xi-1}\exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)$$

$$= \frac{\xi^2}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}\left[\omega t^{\omega\xi-1}\exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)-\frac{t^{\omega\xi+\xi-1}}{\zeta^{\eta}}\exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)\right]$$

Equating following equation with zero, we get

$$\omega t^{\omega\xi-1}\exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right) = \frac{t^{\omega\xi+\xi-1}}{\zeta^{\eta}}\exp\left(-\frac{t^{\xi}}{\zeta^{\eta}}\right)$$

by solving following equation, we get $t = (\omega\zeta^{\eta})^{1/\xi}$. It is easy to verify that $\frac{d^2}{dt^2}f(t) < 0$, which resulting $t^* = (\omega\zeta^{\eta})^{1/\xi}$ as mode.

2.3. Quantile Function

Theorem 2.4. If T ~ LBLT(ω,η,ξ,ζ), then the quantile function of T defined by

$$t = \left(\zeta^{\eta}Q^{-1}\left(\omega+\frac{1}{\xi}, 1-U\right)\right)^{1/\xi} \tag{2.14}$$

Proof. $U = F(t)$

$$U = \frac{\gamma\left(\omega+\frac{1}{\xi}, \frac{t^{\xi}}{\zeta^{\eta}}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}$$

Using basic concept of incomplete gamma function, we get

$$1-U = \frac{\Gamma\left(\omega+\frac{1}{\xi}, \frac{t^{\xi}}{\zeta^{\eta}}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}$$

$$1-U = Q\left(\omega+\frac{1}{\xi}, \frac{t^{\xi}}{\zeta^{\eta}}\right)$$

where, $Q(a,z)$ is regularized incomplete gamma function.

Using basic of inverse regularized incomplete gamma function

$$\frac{t^{\xi}}{\zeta^{\eta}} = Q^{-1}\left(\omega+\frac{1}{\xi}, 1-U\right)$$

theorem follows by solving the following equation.

2.4. Order Statistics

Let for the random samples T_1, T_2, ..., T_n taken from continuous population with pdf $f_T(t)$ and cdf with $F_T(t)$, thus $T_{(1)}, T_{(2)}, ..., T_{(n)}$ be the order statistics, then the pdf and cdf of r^{th} order statistics $T_{(r)}$ is given by

$$f_{T_{(r)}}(t) = \frac{n!}{(r-1)!(n-r)!}[F_T(t)]^{r-1}[1-F_T(t)]^{n-r}f_T(t) \tag{2.15}$$

using equation (2.1) and (2.2) in equation (2.15), we get the pdf of r^{th} order statistics $T_{(r)}$ of LBLT, given by

$$f_{T_{(r)}}(t) = \frac{n!}{(r-1)!(n-r)!}\left[\frac{\gamma\left(\omega+\frac{1}{\xi},\ \frac{t^{\xi}}{\zeta\eta}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}\right]^{r-1}\left[\frac{\Gamma\left(\omega+\frac{1}{\xi},\ \frac{t^{\xi}}{\zeta\eta}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}\right]^{n-r}\frac{\xi t^{\omega\xi}\exp\left(-\frac{t^{\xi}}{\zeta\eta}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}$$

$$= \frac{n!}{(r-1)!(n-r)!}\frac{\xi t^{\omega\xi}\ \exp\left(-\frac{t^{\xi}}{\zeta\eta}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\left(\Gamma\left(\omega+\frac{1}{\xi}\right)\right)^{n}}\left[\gamma\left(\omega+\frac{1}{\xi},\ \frac{t^{\xi}}{\zeta\eta}\right)\right]^{r-1}\left[\Gamma\left(\omega+\frac{1}{\xi},\ \frac{t^{\xi}}{\zeta\eta}\right)\right]^{n-r} \quad (2.16)$$

by using equation (2.16), we get pdf of 1^{st} and n^{th} order statistics respectively,

$$f_{T_{(1)}}(t) = \frac{n\,\xi t^{\omega\xi}\ \exp\left(-\frac{t^{\xi}}{\zeta\eta}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\left(\Gamma\left(\omega+\frac{1}{\xi}\right)\right)^{n}}\left[\Gamma\left(\omega+\frac{1}{\xi},\ \frac{t^{\xi}}{\zeta\eta}\right)\right]^{n-1}$$

$$f_{T_{(n)}}(t) = \frac{n\,\xi t^{\omega\xi}\ \exp\left(-\frac{t^{\xi}}{\zeta\eta}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\left(\Gamma\left(\omega+\frac{1}{\xi}\right)\right)^{n}}\left[\gamma\left(\omega+\frac{1}{\xi},\ \frac{t^{\xi}}{\zeta\eta}\right)\right]^{n-1}$$

3. Parameters Estimation

In present section, we will estimate the parameters of the LBLT distribution using the maximum likelihood technique. Let $t_1, t_2, \ldots, t_n$ be the random samples of size n follows the LBLT distribution, then the likelihood function

$$L = \frac{\xi^{n}}{\zeta^{n\eta\left(\omega+\frac{1}{\xi}\right)}\left(\Gamma\left(\omega+\frac{1}{\xi}\right)\right)^{n}}\exp\left(-\frac{\sum t^{\xi}}{\zeta\eta}\right)\prod_{i=1}^{n} t^{\omega\xi}$$

The log-likelihood function can be written as

$$\log L = n\log\xi - n\eta\left(\omega+\frac{1}{\xi}\right)\log\zeta - n\log\Gamma\left(\omega+\frac{1}{\xi}\right) - \frac{1}{\zeta\eta}\sum_{i=1}^{n} t_i^{\xi} + \omega\xi\sum_{i=1}^{n} t_i \quad (3.1)$$

differentiating equation (3.1), with respect to ω, η, ξ and ζ equating with zero respectively, we get normal equations

$$\frac{d\log L}{d\omega} = -n\eta\log\zeta - n\psi\left(\omega+\frac{1}{\xi}\right) + \xi\sum_{i=1}^{n} t_i \quad (3.2)$$

$$\frac{d\log L}{d\eta} = -n\left(\omega+\frac{1}{\xi}\right)\log\zeta + \frac{1}{\zeta\eta}\log\zeta\sum_{i=1}^{n} t_i^{\xi} \quad (3.3)$$

$$\frac{d\log L}{d\xi} = \frac{n}{\xi} + \frac{n\eta\log\zeta}{\xi^2} - n\psi\left(\omega+\frac{1}{\xi}\right) - \frac{1}{\zeta\eta}\sum_{i=1}^{n} t_i^{\xi}\log t_i + \omega\sum_{i=1}^{n} t_i \quad (3.4)$$

$$\frac{d\log L}{d\zeta} = -\frac{n\eta}{\zeta}\left(\omega+\frac{1}{\xi}\right) + \frac{\eta}{\zeta\eta+1}\sum_{i=1}^{n} t_i^{\xi} \quad (3.5)$$

where, $\psi(z) = \frac{d}{dz}\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is a logarithmic derivative of gamma function. As it seems, from equations (3.2), (3.3), (3.4) and (3.5), the analytical solution of ω, η, ξ and ζ are not available. Consequently, Iterative methods must be used for non-linear parameter estimation.

4. Entropy, Bonferroni and Lorenz curves

Entropy is an important concept in several academic disciplines, including physics, communication theory, probability and statistics, and economics. To quantify the degree to which a system exhibits diversity, uncertainty, or unpredictability, one might use the entropy metric. A random variable's entropy You may think of T as a quantitative measure of the amount of variance or uncertainty linked to it.

4.1. Shannon's Entropy

Shannon's entropy is defined as

$$S(t) = -E(\log f(t))$$

$$= -E\left(\log\left(\frac{\xi t^{\omega\xi}\exp\left(-\frac{t^{\xi}}{\zeta\eta}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}\right)\right)$$

$$= -E\left(\log\left(\frac{\xi}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}\right) - \frac{t^{\xi}}{\zeta^{\eta}} + \omega\xi\log(t)\right)$$

$$= -\log\left(\frac{\xi}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}\right) + \frac{E(t^{\xi})}{\zeta^{\eta}} - \omega\xi\,E(\log(t)) \quad (4.1)$$

we get

$$E(t^{\xi}) = \zeta^{\eta}\left(\omega+\frac{1}{\xi}\right) \quad (4.2)$$

$$E(\log(t)) = \frac{1}{\xi}\psi\left(\omega+\frac{1}{\xi}\right) + \frac{\eta}{\xi}\log\zeta \quad (4.3)$$

using equation (4.2) and (4.3) in equation (4.1), we get

$$S(t) = -\log\left(\frac{\xi}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}\right) + \left(\omega+\frac{1}{\xi}\right) - \omega\left(\psi\left(\omega+\frac{1}{\xi}\right) + \eta\log\zeta\right) \quad (4.4)$$

4.2. Renyi Entropy

Renyi entropy of order k is defined as

$$H_k = \frac{1}{1-k}\log\int_0^{\infty}(f(t))^k dt$$

$$= \frac{1}{1-k}\log\left(\int_0^{\infty}\left(\frac{\xi t^{\omega\xi}\exp\left(-\frac{t^{\xi}}{\zeta\eta}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}\right)^{k} dt\right)$$

$$= \frac{1}{1-k}\log\left(\int_0^{\infty}\left(\frac{\xi}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}\right)^{k} t^{\omega\xi k}\exp\left(-\frac{kt^{\xi}}{\zeta\eta}\right)dt\right)$$

by using transformation $y = \left(\frac{kt^{\xi}}{\zeta\eta}\right)$, we get

$$H_k = \frac{1}{1-k}\log\left(\frac{\xi^{k-1}\Gamma\left(\omega k+\frac{1}{\xi}\right)}{\zeta^{\frac{\eta}{\xi}(k-1)}k^{\omega k+\frac{1}{\xi}}\left(\Gamma\left(\omega+\frac{1}{\xi}\right)\right)^{k}}\right) \quad (4.5)$$

4.3. Lorenz curves and Bonferroni

It is assumed that the random variable T is non-negative and has a twice-differentiable, continuous cumulative distribution function. The Bonferroni curve for the random variable T is defined as follows:

$$B(p) = \frac{1}{p\mu}\int_0^q tf(t)dt$$

where, $p = F(t)$, $q = F^{-1}(p)$ and $\mu = E(t)$

$$= \frac{1}{p\left(\zeta^{\eta/\xi}\frac{\Gamma\left(\omega+\frac{2}{\xi}\right)}{\Gamma\left(\omega+\frac{1}{\xi}\right)}\right)}\int_0^q \frac{\xi t^{\omega\xi+1}\exp\left(-\frac{t^\xi}{\zeta^\eta}\right)}{\zeta^{\eta\left(\omega+\frac{1}{\xi}\right)}\Gamma\left(\omega+\frac{1}{\xi}\right)}dt$$

$$= \frac{\xi}{p\zeta^{\omega\eta}\Gamma\left(\omega+\frac{2}{\xi}\right)}\int_0^q t^{\omega\xi+1}\exp\left(-\frac{t^\xi}{\zeta^\eta}\right)dt$$

by using transformation $y = \left(\frac{t^\xi}{\zeta^\eta}\right)$, we get

$$B(p) = \frac{\zeta^{2\eta/\xi}}{p\Gamma\left(\omega+\frac{2}{\xi}\right)}\gamma\left(\omega+\frac{2}{\xi}, \frac{q^\xi}{\zeta^\eta}\right) \quad (4.6)$$

Lorenz curve is defined as

$$L(p) = \frac{1}{\mu}\int_0^q tf(t)dt$$

$$= pB(p)$$

by using equation (4.6), we get

$$L(p) = \frac{\zeta^{2\eta/\xi}}{\Gamma\left(\omega+\frac{2}{\xi}\right)}\gamma\left(\omega+\frac{2}{\xi}, \frac{q^\xi}{\zeta^\eta}\right) \quad (4.7)$$

5. Real data Application

The Length-biased Lifetime (LBLT) distribution is applied on two actual lifetime datasets. The results indicate that both datasets exhibit a superior match with the Length-biased Lifetime (LBLT) distribution as compared to the Length biased weighted Lindley distribution, Length-Biased Gamma-Rayleigh Distribution and Two-Parameter Weighted Rama Distribution.

The first set of data shows how long it took for twenty people to feel better after taking an analgesic. Sule and Halid (2023) [17] have utilized this data set.

The summary of data set is given as:

Min.	Max	Mean	S.D.	S.E.	Median	Mode
1.1	4.1	1.9	0.7041	0.1574	1.7	1.7

Table 2: The ML estimates for first data set

Distribution	ω	η	ξ	ζ
LBLT	79.4897	3.2270	0.3464	0.2721
LBWL			1.9011	1.53E-05
LBGR			1.8478	0.5753
TPWR			0.4251	1.0770

Table 3: Evaluate the first dataset's goodness of fit

Distribution	– logL	AIC	BIC	AICC	HQIC
LBLT	17.096	42.193	46.176	44.546	42.971
LBWL	22.104	48.209	50.201	48.840	48.597
LBGR	19.170	42.340	44.332	42.972	42.729
TPWR	24.302	52.603	54.595	53.235	52.992

where,

$$AIC = 2P - 2\log L$$
$$BIC = P\log n - 2\log L$$
$$AICC = AIC + \frac{2P(P+1)}{n-P-1}$$
$$HQIC = 2P\log(\log(n)) - 2\log L$$

n = sample size, P = number of parameters.

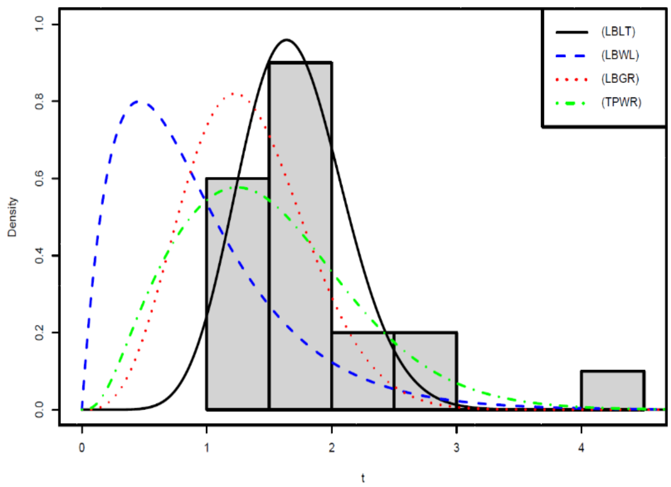

Figure 4: Data fitting plots with densities for first data set

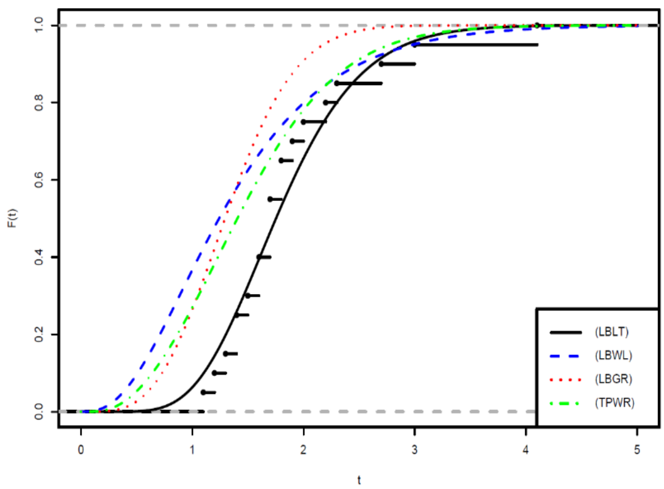

Figure 5: Data fitting plots with ecdf for first data set

The durability of 1.5 cm glass fiber tested at the National Physical Laboratory in England is represented by the second set of data used by Alzaatreh *et. al* (2015) [18]. Summary of the second data set

Min.	Max.	Mean	S.D.	S.E.	Median	Mode
0.550	2.24	1.523	0.3153	0.0397	1.6	1.61

Table 4: The ML estimates for second data set

Distribution	ω	η	ξ	ζ
LBLT	0.5237	2.6237	7.7951	5.4140226
LBWL			6.0702	1.09E-06
LBGR			4.908438	2.238172
TPWR			0.6942377	1.455

Table 5: Evaluate the second dataset's goodness of fit

Distribution	− log L	AIC	BIC	AICC	HQIC
LBLT	12.860	33.719	42.292	34.3859	37.09
LBWL	203.108	410.215	414.502	410.409	411.9
LBGR	19.282	42.564	46.850	42.757	44.25
TPWR	24.302	52.604	54.595	53.235	52.99

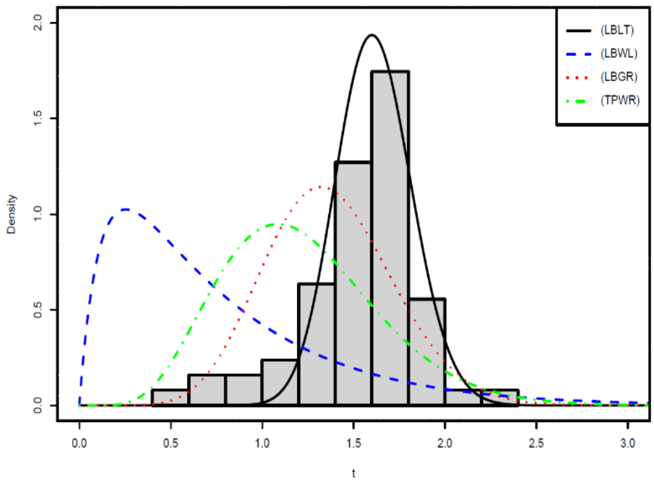

Figure 6: Data fitting plots with densities for second data set

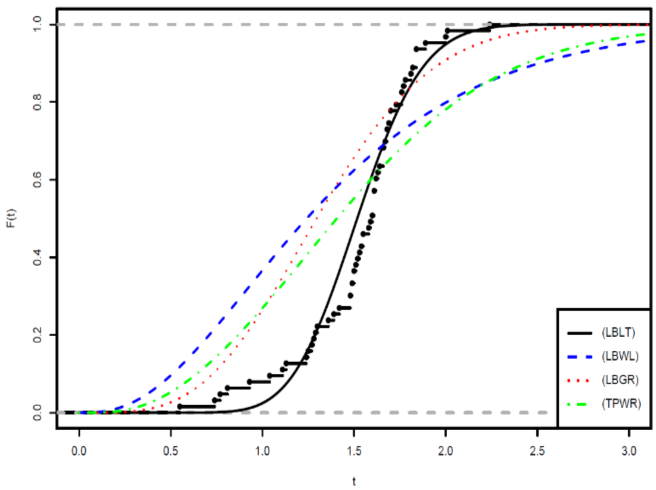

Figure 7: Data fitting plots with ecdf for second data set

It is easy to notice from Table 3, 5 and Fig. 4, 5, 6 and 7, that the New Length-biased Lifetime distribution exhibits a better fit as compared to the Length biased weighted Lindley distribution, Length-Biased Gamma-Rayleigh distribution and Two-Parameter Weighted Rama distribution for fitting the datasets.

6. Conclusion

The Length-biased Lifetime distribution is a new distribution that is suggested in this article. Four parameters, shape and scale parameters, define the distribution under consideration. Parameter estimation and characteristics of this distribution, including its moments, failure rate, reliability function, etc., are thoroughly investigated using specific formulas. We have analysed and compared the AIC, BIC, AICC, and HQIC criteria. Both data sets from real life shows that AIC is minimum for LBLT as shown in table 3 & 5, and LBLT is better fitted as shown in fig 4, 5, 6 & 7, compared to the Length biased weighted Lindley distribution, Length-Biased Gamma-Rayleigh Distribution and Two Parameter Weighted Rama Distribution.

Data Availability
None

Conflict of Interest
The authors state that they have no competing interests.

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