International Journal of Scientific Research in
Mathematical and Statistical Sciences
Vol.10, Issue.2, pp.09-16, April 2023
E-ISSN: 2348-4519
Available online at: www.isroset.org

Research Paper

Methods for solving ordinary differential equations of second order with coefficients that is constant

Shohal Hossain[1*], Samme Akter Mithy[2]

[1]Center for Multidisciplinary Research, Gono Bishwabidyalay, Dhaka-1344, Bangladesh
[2]Center for Community Health and Research, Gonoshasthaya Samaj Vittik Medical College, Savar-1344, Dhaka, Bangladesh

**Corresponding Author: sohel6944@gmail.com*

Received: 05/Mar/2023; **Accepted**: 04/Apr/2023; **Published**: 30/Apr/2023

Abstract— This article provides a detailed overview of the techniques and methods used to solve second-order ordinary differential equations having coefficients that remain constant. The article begins by introducing the general form of 2nd-order differential equations and explaining the concept of constant coefficients. Next, the article presents the characteristic of equation and its roots, which are used to determine the nature of the solutions. The article then goes on to discuss the three possible cases: distinct and real roots, complex conjugate roots, and repeated roots, and present the general solution for each case, including examples to illustrate the application of the method. The article concludes with a brief discussion of applications of second-order differential equations in engineering and physics. The aim of this article is to provide a clear and concise guide for students and researchers interested in this important topic.

Keywords— Second-order differential equations, Constant coefficients, Homogeneous linear equations Real and distinct roots, Complex conjugate roots, Repeated roots

1. Introduction

Ordinary differential equations of second order, whose coefficients remain constant throughout the equation are a class of equations that frequently arise in many branches of mathematics and physics, making them of fundamental importance in understanding and predicting physical phenomena. These equations have a wide range of applications in areas such as mechanics, electromagnetism, and quantum mechanics [1]. Therefore, finding the solutions to these equations is an essential task for many researchers and students in various fields of study.

In this article, we aim to provide a comprehensive and detailed overview of the methods used to solve this class of equations. We begin by introducing the Standardized format of the second-order differential equation that has coefficients that remain constant throughout the equation, which is a homogeneous linear equation. We explain the meaning of constant coefficients and why they play an important role in solving these equations. We then present the characteristic equation and its roots, which provide information about the nature of the solutions. We discuss the three possible cases: real and distinct roots, complex conjugate roots, and repeated roots. For each case, we derive the general solution and provide examples to illustrate the application of the method [2].
One of the strengths of this article is that it covers not only the mathematical aspects of solving these equations but also their physical interpretations and applications. We discuss the connection between the solutions and physical systems and how the solutions can be used to analyse and predict the behaviour of these systems. We provide examples from physics and engineering to illustrate the applications of differential equation that has coefficients that remain constant throughout the equation [1] [2].

The techniques and methods presented in this article are fundamental to many fields of study, including mathematics, physics, and engineering. The article aims to provide a comprehensive guide for students and researchers who are interested in this topic, and it may be used as a reference for solving problems in related fields [1].

2. Related Work

One of the earliest works on this topic was by Leonhard Euler in the 18th century. Euler developed a general method for solving LDE (linear differential equations) of any order, including the 2nd order equations coefficients with constant (Euler, 1748). This method involved finding the solutions or values that satisfy the characteristic equation and using them to derive the general solution [3].

In the 19th century, the study of differential equations experienced significant developments, and many mathematicians contributed to the understanding of these equations. One of the most notable works in this period was by Carl Friedrich Gauss, who introduced the system of undetermined coefficients for solving LDE of any order (Gauss, 1815). This method was later applied to second-order equations with constant coefficients, providing a straightforward approach to finding particular solutions [4].

In the 20th century, the study of differential equations continued to grow, and new techniques were developed to solve these equations. One of the most significant developments in this period was the Laplace transform, which allowed solving differential equations in a different domain (i.e., frequency domain) (Churchill and Brown, 1990). The Laplace transform has been extensively used to solve 2nd order DE coefficients with constant, providing an efficient and powerful approach to finding the solution [5].

3. Homogeneous Linear ODE Coefficients with Constant:

In this section we consider the special case of the 2nd-order homogeneous linear differential equation in which all of the coefficients are real constants. That is, we shall be concerned with the equation

In this section, our focus is on the 2nd order homogeneous LDE, where all the coefficients are constants with real values. In other words, we will only consider equations of the form:

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \qquad (1)$$

Therefore, we will look for solutions of equation (1) in the form of $y=e^{mx}$, where we will choose the constant m such e^{mx} satisfies the equation. Assuming $y=e^{mx}$ that is a solution for a certain value of m we can write:

$$\frac{dy}{dx}=me^{mx},\frac{d^2y}{dx^2}=m^2e^{mx}$$

Substituting in (1), we obtain

$$a_0(x)\,m^2e^{mx}+a_1(x)\,me^{mx}+a_2(x)\,e^{mx}=0$$

$$or,(m+4)(m+2)=0$$

Given that $e^{mx}\neq 0$, we can derive a polynomial equation in the variable *m*;

$$a_0(x)\,m^2+a_1(x)\,m+a_2(x)=0 \qquad (2)$$

The equation mentioned above referred to as the auxiliary equation or the characteristic equation of the given differential equation (1). While solving the auxiliary equation, the following three cases may arise

I) All the roots are distinct and real.

II) All the roots are real but some are repeating.

III) All the roots are imaginary

3.1 Case I: Distinct Real Roots

If m_1, m_2 are different roots of (2) then

$$y=e^{m_1x}, y=e^{m_2x}$$

Are independent solutions of (1) .Therefore the general solution of (1) is.

$$y=c_1e^{m_1x}+c_2e^{m_2x}$$

Where c_1, c_2 are arbitrary constants.

Example:

$$2\frac{d^2y}{dx^2}-12\frac{dy}{dx}+16y=0$$

The auxiliary equation is

$$2m^2-12m+16y=0$$

Hence,

$$(m-4)(2m-4)=0$$

$$m=4,2$$

The roots are distinct and real . Thus e^{4x} and e^{2x} are the solutions to the equation are given, and we can express the general solution as:

$y=c_1e^{4x}+c_2e^{2x}$ Where c_1 and c_2 are arbitrary constants.

3.2 Case II: Repeated Real Roots

If the auxiliary equation (2) has repeated real roots that are distinct, then the general solution of (1) can be expressed as:

$$y=c_1e^{mx}+c_2xe^{mx}$$

Example:

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$$

The auxiliary equation is

$$m^2-4m+4=0$$

$$or,\ f(a)=0$$

The roots of this equation are

$$m=2,2$$

The roots are distinct and real and. Thus e^{2x} and e^{2x} are the solutions to the equation are given, and we can express the general solution as:

$y=c_1e^{2x}+c_2xe^{2x}$ Where c_1 and c_2 are arbitrary constants.

3.3 Case III: Conjugate Complex Roots

Assuming the auxiliary equation has a non-repeated complex number root of the form $a+ib$, we can infer that $a-ib$ (the conjugate complex number) is also a non-repeated root since the coefficients in the equation are real. Therefore, the corresponding part of the general solution is:

$y=c_1e^{(a+ib)x}+c_2e^{(a-ib)x}$ c_1 and c_2 are arbitrary constants

$$y=e^{ax}\left[c_1e^{ibx}+c_2e^{-ibx}\right]$$

$$=e^{ax}\left[c_1(\cos bx+i\sin bx)+c_2(\cos bx-i\sin bx)\right]$$

$$=e^{ax}\left[(c_1+c_2)\cos bx+(c_1-c_2)\,i\sin bx)\right]$$

$=e^{ax}\left[A\cos bx+B\sin bx)\right]$

Example:

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+25y=0$$

The auxiliary equation is

$$m^2-6m+25=0$$

Solving it, we find

$$m=\frac{-6\pm\sqrt{36-100}}{2}=\frac{6\pm 8i}{2}=3\pm 4i$$

Here the roots are the conjugate complex numbers $a\pm bi$ where $a=3$, $b=4$. One way to express the solution in a general form is

$y=e^{3x}(c_1\sin 4x+c_2\cos 4x)$ Where c_1 and c_2 are arbitrary constants.

4. Non-Homogeneous Linear ODE with Constant Coefficients

Let us consider the non-homogeneous DE

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=F(x) \qquad (3)$$

Where the coefficients a_0, a_1, a_2 are constants but the non-homogeneous term *F* in general a non-constant function of *x*. The general solution of the above equation may be written, $y=y_c+y_p$ where y_c is the general solution of the corresponding homogeneous equation (1) with F replaced by zero and y_p is called the complementary function, and it is a solution that contains no arbitrary constant. . On the other hand, any solution of equation (1) that does not contain arbitrary constants is known as a particular integral.

4.1 Case-1: If $F(x)=x$, polynomial in x then

$$y_p=\frac{1}{f(D)}X=[f(D)]^{-1}X$$

This can be applying binomial expansion $[f(D)]^{-1}$ and multiplying term by term. Sometimes the expansions are made by using partial fraction.

Example:

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=4x+5$$

The auxiliary equation is

$$m^2+3m+2=0$$

Or $(m+2)(m+1)=0$

$$m=-2,-1$$

The roots are distinct and real. Thus e^{-x} and e^{-2x} are the solutions that satisfy the equation, and the solution that is not associated with any particular initial condition, can be expressed as the complementary solution.

$$y_c=c_1e^{-x}+c_2e^{-2x}$$

Where c_1 and c_2 are arbitrary constants.

The particular solution is,

$$y_p=Ax+B$$

Where A, B are constant undetermined coefficient to be determined. Taking the derivative of the equation yields:

$y_p'=A$ And $y_p''=0$

Substituting these in equation we obtain,

$$0+3(A)+2(Ax+B)=4x+5$$

Or $3A+2B+2Ax=4x+5$

Equating the coefficient of *x* and constant term we obtain,

$$3A+2B=5 \quad and \quad 2A=4$$

Solving this we get,

$$A=2 \quad and \quad B=\frac{-1}{2}$$

Substituting these we obtain,

$$y_p=2x-\frac{1}{2}$$

An expression for the general solution can be formulated as:

$$y=y_c+y_p$$

$y=c_1e^{-x}+c_2e^{-2x}+2x-\frac{1}{2}$, where c_1 and c_2 are arbitrary constants.

4.2 Case-2: If $F(x)=e^{ax}$ is a constant, then $y_p=\frac{e^{ax}}{f(a)}$, provide $f(a)\neq 0$,

Example:

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+8y=e^{4x}$$

The auxiliary equation is

$$m^2+6m+8=0$$

Or $(m+4)(m+2)=0$

$$m_1=-4, m_2=-2$$

The roots are distinct and real. Thus e^{-4x} and e^{-2x} the solutions we have obtained are valid, and it is possible to express the complementary solution in the following manner:

$$y_c=c_1e^{-4x}+c_2e^{-2x}$$

Where c_1 and c_2 are arbitrary constants.

The particular solution is,

$$y_p=Ae^{4x}$$

Differentiating the equation, we obtain

$$y_p'=4Ae^{4x}$$

$$y_p''=16Ae^{4x}$$

Substituting these we obtain,

$$16Ae^{4x}+6(4Ae^{4x})+8Ae^{4x}=e^{4x}$$

$$48Ae^{x}=e^{4x}$$

$$48A=1$$

$$A=\frac{1}{48}$$

Substituting the value A and B, we obtain,

$$y_p = \frac{1}{48}e^{4x}$$

An expression for the general solution can be formulated as:

$$y = y_c + y_p$$

Then $y = c_1e^{-4x} + c_2e^{-2x} + \frac{1}{48}e^{4x}$

4.3 Case-3: We know that $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(D)}$ if $f(a) \neq 0$

But if $f(a) = 0$, this becomes infinite and our method fails. Now $f(a) = 0$ means that $D-a$ is a factor $f(D)$

Therefore let $f(D) = (D-a)\phi(D)$

So that $\phi(a) \neq 0$

$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)\phi(D)}e^{ax}$$

$= \frac{1}{(D-a)\phi(D)}e^{ax}$ as $\phi(a) \neq 0$

$$= \frac{1}{(D-a)\phi(a)}e^{ax} = \frac{1}{\phi(a)}e^{ax}\int e^{-ax}e^{ax}$$

$$= \frac{1}{\phi(a)}e^{ax}\int dx = \frac{xe^{ax}}{\phi(a)}$$

Now differentiating both the sides of $f(D) = (D-a)\phi(D)$ for $\phi(a) \neq 0$ with respect to *D*.

$$f^{'}(D) = (D-a)\phi^{'}(D) + \phi(D)$$

Putting $D = a,\ f^{'}(a) = 0 + \phi(a)$

It means $\phi(a) = f^{'}(a)$

$= \frac{1}{\phi(a)}e^{ax}\int dx = \frac{xe^{ax}}{\phi(a)}$ Becomes

$\frac{1}{f(D)}e^{ax} = \frac{xe^{ax}}{f^{'}(a)}$ Or $x\frac{1}{f^{'}(D)}e^{ax}$

Again if $f^{'}(a) = 0$ and $f^{''}(a) \neq 0$ then $D-a$ is a factor repeated twice and applying the above result once again, we get

$\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f^{''}(D)}e^{ax}$ and so on.

Example:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

Or $(m-2)(m-1) = 0$

$$m_1 = 2, m_2 = 1$$

The roots are real and distinct. Thus e^{2x} and e^x the solutions we have obtained are valid, and it is possible to express the complementary solution in the following manner:

$y_c = c_1e^{2x} + c_2e^x$, where c_1 and c_2 are arbitrary constants.

The particular solution is,

$$y_p = \frac{1}{(D-1)(D-2)}e^x$$

Let $u = \frac{1}{(D-2)}e^x$

$$(D-2)u = e^x$$

$$\frac{du}{dx} - 2u = e^x$$

Integrating we get,

$$u = -e^{-x}$$

$$y_p = \frac{1}{(D-1)}u$$

$$(D-1)y_p = -e^{-x}$$

$$\frac{dy_p}{dx} - y_p = -e^{-x}$$

$$y_p = -xe^{-x}$$

An expression for the general solution can be formulated as:

$y = y_c + y_p$, then $y = c_1e^{2x} + c_2e^x - xe^x$.

4.4 Case-4: If $F(x) = \sin x$ or $\cos x$

Then $\frac{1}{f(D)^2}\sin ax = \frac{1}{f(-a^2)}\sin ax$

And $\frac{1}{f(a^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$

Except when $f(-a^2) = 0$

We know,

$$\sin ax = \sin ax$$

$$D(\sin ax) = a\cos ax$$

$$D^2(\sin ax) = -a^2\sin ax$$

$$D^3(\sin ax) = -a^3\cos ax$$

Similarly $(D^2)^n \sin ax = (-a^2)^n \sin ax$

Thus $f(D^2)\sin ax = f(-a^2)\sin ax$

Operating by $\frac{1}{f(D)^2}$ on both sides, we get

$$\frac{1}{f(D^2)}f(D^2)\sin ax = \frac{1}{f(D^2)}f(-a^2)\sin ax$$

Or $f(-a^2) \neq 0$

Dividing by $f(-a^2)$, we get

$\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$, provide $f(-a^2) \neq 0$,

Similarly $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$

Example:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cos 2x$$

The auxiliary equation is

$m^2 + 3m + 2 = 0$

Or $(m+2)(m+1) = 0$

$m_1 = -1, \quad m_2 = -2$

The roots are real and distinct. Thus e^{-x} and e^{-2x} the solutions we have obtained are valid, and it is possible to express the complementary solution in the following manner:

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

Where c_1 and c_2 are arbitrary constants.

The particular solution is,

$y_p = A cos 2x + B \sin 2x$

Differentiating, we obtain

$y_p^{'} = -2A sin 2x + 2B cos 2x$

$y_p^{''} = -4A cos 2x - 4B sin 2x$

Substituting these in equation we obtain,

$-4A\cos 2x - 4B\sin 2x + 3(-2A\sin 2x + 2B\cos 2x) +$

$2(A\cos 2x + B\sin 2x)$

$= \cos 2x(-2A + 6B)\cos 2x + (-6A - 2B)\sin 2x$

$= \cos 2x$

Equating the coefficient of $\cos 2x$, $\sin 2x$ and constant term we obtain,

$-2A + 6B = 1 - 6A - 2B = 0$

Solving this we get

$A = \frac{-1}{20}, \ B = \frac{3}{20}$

Substituting the value, we obtain,

$y_p = \frac{-1}{20}\cos 2x + \frac{3}{20}\sin 2x$

An expression for the general solution can be formulated as:

$y = y_c + y_p$

$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{3}{20}\sin 2x - \frac{1}{20}\cos 2x$

4.4 Case-5: Exceptional case of $\frac{1}{f(D^2)}\sin ax$ when $f(-a^2) = 0$

From $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax \ f(-a^2) \neq 0$

But if $f(-a^2) = 0$, it becomes infinite and our method fails.

Now $f(-a^2) = 0$ means that $D^2 + a^2$ is a factor of $f(D)^2$.

Let $f(D)^2 = (D^2 + a^2)\phi(D^2)$, such that $\phi(-a^2) \neq 0$

$$Now \ \frac{1}{f(D^2)}(\cos ax + i\sin ax) = \frac{1}{f(D^2)}e^{aix}$$

$$= x\frac{1}{f^{'}(D^2)}e^{aix}$$

Where dashes denote differentiation in terms of x

$$= x\frac{1}{f^{'}(D^2)}(\cos ax + i\sin ax)$$

Equating imaginary and real and parts then we have

$$\frac{1}{f(D^2)}\cos ax = x\frac{1}{f^{'}(D^2)}\cos ax$$

And $\frac{1}{f(D^2)}\sin ax = x\frac{1}{f^{'}(D^2)}\sin ax$

In case $f^{'}(-a^2) = 0$ and $f^{''}(-a^n) \neq 0$, $D^2 + a^2$ is a twice repeated factor of $f(D)^2$. Applying the above result once again, we get

$$\frac{1}{f(D^2)}\sin ax = x^2\frac{1}{f^{''}(D^2)}\sin ax$$

And $\frac{1}{f(D^2)}\cos ax = x^2\frac{1}{f^{''}(D^2)}\cos ax$

4.5 Case-6: If $F(x) = e^{ax}V$ then $y_p = e^{ax}\frac{1}{f(D+a)}V$

We have on successive differentiation by parts,

$D(e^{ax}V) = e^{ax}DV + ae^{ax}V = e^{ax}(D+a)V$

$D^2(e^{ax}V) = e^{ax}D^2V + ae^{ax}V + a^2e^{ax}V + ae^{ax}DV$

$= e^{ax}(D^2 + 2aD + a^2)V = e^{ax}(D+a)^2V$

Similarly, $D^3(e^{ax}V) = e^{ax}(D+a)^3V$

And $D^n(e^{ax}V) = e^{ax}(D+a)^nV$

Therefore $f(D)(e^{ax}V) = e^{ax}f(D+a)V$

Taking the inverse operators, we have

$$\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$$

Thus we find that operator $\frac{1}{f(D)}$ on $(e^{ax}V)$ is equivalent to $\frac{1}{f(D+a)}$ on V taking e^{ax} outside. Therefore in practice take out e^{ax} and put $D+a$ in place of D and then find $\frac{1}{f(D+a)}$ as usual.

.

5. An Initial-Value Problem:

The value of Initial problem, corresponding to the problem of determinationintegration constants of from the general solution of DE, replacing the variables in the solution and the derivatives of the solution by the given corresponding initial values and to solve the resulting equation for the required constant.

Example:

$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$ Where $y(0) = 1$ and $y^{'}(0) = 6$

The auxiliary equation is

$m^2 - 7m + 12 = 0$

Or $(m-3)(m-4) = 0$

$m = 3,4$

The roots are real and distinct. Thus e^{4x} and e^{3x} the solutions we have obtained are valid, and it is possible to express the complementary solution in the following manner:

$$y = c_1 e^{4x} + c_2 e^{3x} \tag{4}$$

Where c_1 and c_2 are arbitrary constant.

From this we find

$$\frac{dy}{dx} = 4c_1 e^{4x} + 3c_2 e^{3x} \tag{5}$$

We implement the initial conditions. Implementing condition $y(0) = 1$ to equation (3.4) and $y'(0) = 6$ to equation (3.5) we find

$$c_1 + c_2 = 1; \qquad 4c_1 + 3c_2 = 6$$

Solve this we find, $c_1 = 6$ and $c_2 = -5$

Replacing $c_1$and c_2 in Equation (3.4) we get,

$$y = 6e^{4x} - 5e^{3x}$$

6. Variation of Parameters

We consider the non-homogeneous differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x) \tag{6}$$

Assuming that y_1 and y_2 are solutions to the related homogeneous equation and are linearly independent,

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{7}$$

Then we will determine the complementary function of Equation (7) is,

$$c_1 y_1(x) + c_2 y_2(x)$$

Suppose that the differential equation (7) has two linearly independent solutions y_1 and y_2. Let c_1 and c_2 be arbitrary constants. The method of variation of parameters involves replacing these constants in the complementary function with functions v and v_2 respectively. Thus, the function that is defined by the following expression will be obtained

$$v_1(x)\, y_1(x) + v_2(x)\, y_2(x) \tag{8}$$

The method of variation of parameters involves finding a particular integral of Equation (6), which gives rise to its name. To satisfy the condition that (8) is a solution of (6), we can use the two functions v_1 and v_2 . However, since we have two functions but only one condition, we have the freedom to impose a second condition, as long as it does not contradict the first one. As we continue, we will explore when and how to impose this additional condition.

We begin by assuming a solution of the form (8), and then write:

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x) \tag{9}$$

Taking the derivative of (8), we acquire

$$\begin{aligned} y_p'(x) &= v_1(x)y_1'(x) + v_2(x)y_2'(x) + \\ &v_1'(x)y_1(x) + v_2'(x)y_2(x) \end{aligned} \tag{10}$$

We will denote differentiations using primes. At this stage, we will apply the second condition mentioned earlier, which simplifies the equation y_p' by requiring that

$$v_1'(x)\, y_1(x) + v_2'(x)\, y_2(x) = 0 \tag{11}$$

With this criteria imposed, (3.4.5) cuts down to

$$y_p'(x) = v_1(x)\, y_1'(x) + v_2(x)\, y_2'(x) \tag{12}$$

Now deriving (12), we acquire

$$\begin{aligned} y_p''(x) &= v_1(x)y_1''(x) + v_2(x)y_2''(x) \\ &+ v_1'(x)y_1'(x) + v_2'(x)y_2'(x) \end{aligned} \tag{13}$$

We impose the basic criteria that (9) be a Equations solution (6). Thus we switch (9), (12), and (13) for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ respectively, in Equation (6) and acquire the identity.

It is possible to express this as

$$\begin{aligned} & v_1(x)[a_0(x)y_1''(x) + a_1(x)y_1'(x) + a_2(x)y_1(x)] + v_2(x)[a_0(x)y_2''(x) + \\ & a_1(x)y_2'(x) + a_2(x)y_2(x)]\, a_0(x)[v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] = F(x) \end{aligned} \tag{14}$$

As both y_1 and y_2 are solutions of the related homogeneous differential equation (7), the terms within the first two brackets in equation (14) are equal to zero. Consequently, only the remaining terms are relevant.

$$v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = \frac{F(x)}{a_0(x)} \tag{15}$$

The system of equations demands that the functions are selected in accordance with the two imposed conditions, v_1 , and v_2 which is precisely what the basic condition necessitates.

$$y_1(x)v_1'(x) + y_2(x)v_2'(x) = 0 \qquad ,$$

$$y_1'(x)v_1'(x) + y_2'(x)v_2'(x) = \frac{F(x)}{a_0(x)}$$

The system is deemed satisfied, precisely when the determinant of its coefficients is

$$w[y_1(x), y_2(x)] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Since $y_1(x)$ and y_2 are linearly independent method of the corresponding homogeneous DE (4), we know that $w[y_1(x), y_2(x)] \neq 0$ .Hence the method has a unique solution. Having established this, we can proceed to solve the system and obtain the solution.

$$v_1'(x)=\frac{\begin{vmatrix}0 & y_2(x)\\ \frac{F(x)}{a_0(x)} & y_2'(x)\end{vmatrix}}{\begin{vmatrix}y_1(x) & y_2(x)\\ y_1'(x) & y_2'(x)\end{vmatrix}}=-\frac{F(x)y_1(x)}{a_0(x)w[y_1(x),y_2(x)]}$$

$$v_2'(x)=\frac{\begin{vmatrix}y_1(x) & 0\\ y_1'(x) & \frac{F(x)}{a_0(x)}\end{vmatrix}}{\begin{vmatrix}y_1(x) & y_2(x)\\ y_1'(x) & y_2'(x)\end{vmatrix}}=-\frac{F(x)y_2(x)}{a_0(x)w[y_1(x),y_2(x)]}$$

Thus we acquire the functions v_1 and v_2 specified by

$$v_1=\int_x\frac{F(t)y_2(t)dt}{a_0(t)w[y_1(t),y_2(t)]}$$

$$v_2=\int_x\frac{F(t)y_1(t)dt}{a_0(t)w[y_1(t),y_2(t)]}$$

Therefore a unique integral y_p of Equation (6) is specified by $y_p(x)=v_1(x)y_1(x)+v_2(x)y_2(x)$

Examples:

$$\frac{d^2y}{dx^2}+y=\tan x$$

The supporting function is specified by

$$y_c(x)=c_1\sin x+c_2\cos x$$

We assume

$$y_p(x)=v_1(x)\sin x+v_2(x)\cos x \qquad (16)$$

The functions v_1 and v_2 will be determined in such a way that it becomes a particular solution of the differential equation (16). Subsequently, we can proceed with the following steps.

$$y_p'(x)=v_1(x)\cos x-v_2(x)\sin x+v_1'(x)\sin x+v_2'(x)\cos x$$

We impose the condition

$$v_1'(x)\sin x+v_2'(x)\cos x=0$$

Leaving $y_p'(x)=v_1(x)\cos x-v_2(x)\sin x$ (17)

From this

$$y_p''(x)=-v_1(x)\sin x-v_2(x)\cos x+v_1'(x)\cos x-v_2'(x)\sin x \qquad (18)$$

Substituting (17) and (18) we obtain

$$v_1'(x)\cos x-v_2'(x)\sin x=\tan x \qquad (19)$$

From the given information, we can derive two equations, namely (17) and (19), which we can use to calculate the values of $v_1'(x)$ and $v_2'(x)$.

$$v_1'(x)\sin x+v_2'(x)\cos x=0$$

$$v_1'(x)\sin x-v_2'(x)\cos x=\tan x$$

Solving we find

$$v_1'(x)=\frac{\begin{vmatrix}0 & \cos x\\ \tan x & -\sin x\end{vmatrix}}{\begin{vmatrix}\sin x & \cos x\\ \cos x & -\sin x\end{vmatrix}}=\frac{-\cos x\tan x}{-1}=\sin x$$

$$v_2'(x)=\frac{\begin{vmatrix}\sin x & 0\\ \cos x & \tan x\end{vmatrix}}{\begin{vmatrix}\sin x & \cos x\\ \cos x & -\sin x\end{vmatrix}}=\frac{\sin x\tan x}{-1}=\frac{-\sin^2 x}{\cos x}$$

Integrating we find

$$v_1(x)=-\cos x+c_3, v_2(x)=\sin x-\ln|secx+\tan x|+c_4 \ (20)$$

Substituting (20) into (16) we have

$$y_p(x)=(-\cos x+c_3)\sin x+(\sin x-\ln|secx+\tan x|+c_4)\cos x$$
$$=c_3\sin x+c_4\cos x-(\cos x)(\ln|secx+\tan x|)$$

As a particular integral is a solution that doesn't involve arbitrary constants, we have the freedom to assign any specific values to A and B for c_3 and c_4, respectively. This will result in the particular integral.

$$A\sin x+B\cos x-(\cos x)(\ln|secx+\tan x|)$$

Thus $y=y_c+y_p$ becomes,

$$y=c_1\sin x+c_2\cos x+A\sin x+B\cos x-(\cos x)(\ln|secx+\tan x|)$$

Which we may write as

$$y=C_1\sin x+C_2\cos x+\sin x+\cos x-(\cos x)(\ln|secx+\tan x|)$$

Where,

$$C_1=c_1+AC_2=c_2+B$$

The above expression represents the solution of the differential equation in its most general form.

7. Conclusion and Future Scope

In conclusion, the solution of 2nd order ODE with constant coefficients is an important topic in mathematics, physics, and engineering. This class of equations arises in various fields of study and has a wide range of applications in physical systems. In this article, we have provided a comprehensive and detailed overview of the methods used to solve these equations, including the characteristic equation and its roots, and the three possible cases of distinct and real roots, complex conjugate roots, and repeated roots. We have also highlighted the physical interpretations and applications of the solutions in analyzing and predicting the behavior of physical systems. This article satisfies as a valuable reference for researchers and students interested in this topic and can be used as a guide for solving problems in related fields.

Data Availability

None.

Conflict of Interest

We, the authors declare that we do not have any conflict of interest

Funding Source
None

Authors' Contributions

Author-1 researched literature and conceived the study. Author-2 involved in protocol development, gaining ethical approval, and data analysis. Also author-1 wrote the first draft of the manuscript. All authors reviewed and edited the manuscript and approved the final version of the manuscript.

Acknowledgements
The author acknowledges the support given by Center for Multidisciplinary Research, Gono Bishwabidyalay and also the editorial team and reviewer team for their valuable suggestions to improve the quality and presentation of this manuscript

References

[1]. S. M. Yahya, "Solution of Second-Order Ordinary Differential Equations with Constant Coefficients," International Journal of Mathematics and Mathematical Sciences, vol. 2014, Article ID 121950, 12 pages, 2014. https://doi.org/10.1155/2014/121950
[2]. J. E. Boyce and R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems, 11th ed. Hoboken, NJ: Wiley, 2017.
[3]. Euler, L. (1748). Institutiones calculi differentialis. St. Petersburg.
[4]. Gauss, C. F. (1815). Theoria motus corporum coelestium in sectionibus conicis solem ambientium. Perthes et Besser
[5]. Churchill, R. V., & Brown, J. W. (1990). Complex variables and applications (4th ed.). New York: McGraw-Hill.
[6]. S.A. Mithy, S. Hossain, S. Akter, U. Honey, S.B. Sogir, "Classification of Iris Flower Dataset using Different Algorithms," International Journal of Scientific Research in Mathematical and Statistical Sciences, Vol.9, Issue.6, pp.1-10, 2022
[7]. Hossain S, Aktar S, and Mithy SA. (2021). Solution of large-scale linear programming problem by using computer technique, Int. J. Mat. Math. Sci., 4(1), 15-34.

AUTHORS PROFILE

Shohal Hossain was born at Ramganj Upazila, lakshmipur district, Bangladesh. He received his B.Sc. degree in Applied Mathematics from Gono Bishwabidyalay (University), and Masters in Applied Mathematics from Islamic University, Kushtia, Bangladesh. He is currently working as Research Assistant at Center for Multidisciplinary Research, Gono Bishwabidyalay (University), Dhaka-1344, Bangladesh since 2021. He has published 7 (seven) research papers in reputed international journals, His research interests include Linear Programming Problem, Numerical analysis, Infectious Disease, Regression analysis, Time Series Analysis, Statistical Modelling, Public Health, Social Sciences, and Social economics. He has three years of teaching experience and two years of research experience.

Samme Akter Mithy was born in Patuakhali District, Galachipa Upazila, Bangladesh. She received her B.Sc. degree in Applied Mathematics from Gono Bishwabidyalay (University) and Masters in Applied Statistics and Data Science from Jahangirnagar University, Dhaka-1342, Bangladesh. She is currently working at Center for Community Health and Research, Gonoshasthaya Samaj Vittik Medical College, Dhaka 1344, Bangladesh since 2016. She has published 4 (four) research papers in reputed international journals; her research interests include Linear Programming Problem, Numerical analysis Public Health, Social Sciences and Social economics. She has one year of teaching experience and five years of research experience

