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Research Paper

Integral Solution of the Binary Quadratic Equation $3x^2$-y^2=2 that Depicts Parabolas and Hyperbolas

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Abstract: The distinct integral solution of the binary quadratic equation $3x^2$-y^2=2 that depicts distinct parabolas and hyperbolas are examined. A number of non-zero integral solutions are discovered, and some intriguing recurring links between them are provided.

Keywords: Binary quadratic equation, Diophantine equation, Integral solutions, Hyperbola, Parabola & Pell's equation.

1. Introduction

Mathematics is the language of patterns and relationships, and is used to describe anything that can be quantified. One of the biggest and most established subfields in mathematics is number theory. Finding intriguing and surprising relationships is the primary objective of number theory. It is mostly focused on the study of natural numbers and integers. Basic concepts are studied from [1-3].There is a wide range of quadratic Diophantine equations with two unknowns (both homogeneous and non-homogeneous) [4-6].The binary quadratic non-homogeneous equations that represent the various hyperbolas are examined for their non-zero integral solutions in [7–9].

The intriguing binary quadratic equation $3x^2 - y^2 = 2$ presented by an endless number of non-zero integral solutions. The expressed recurring links are fulfilled by the solutions *x* and *y*. Some fascinating characteristics of the arrangements are likewise shown.

2. Basic Definitions

2.1 Diophantine Equation

An equation involving more than one variable in which the Co-efficient of the variables are integer and for which the integer solution are sought.

2.2 Binary Quadratic Diophantine Equation

A general quadratic diophantine equation in two variable x and y is given by, $ax^2 + cy^2 = k$, where a, c and k are satisfied positive or negative values satisfying the equation.

2.3 Pell Equation

A special case of the quadratic diophantine equation having the form
$x^2 - Dy^2 = 1$, where D>0 is a non-square natural number.

3. Analysis Approach

The Diophantine equation represents the binary quadratic equation to be solved for its non-zero distinct integral solution as follows:

$$3x^2 - y^2 = 2 \tag{1}$$

The linear transforms are substituted

$$x = X + T, y = X + 3T \tag{2}$$

in (1), leads to

$$X^2 = 3T^2 + 1 \tag{3}$$

The least possible non negative integer solution to (3) is $(X_0, T_0) = (2,1)$.
The Pell equation (3)'s further solutions can be obtained as

$$\tilde{X}_s + \sqrt{3}\tilde{T}_s = (2+\sqrt{3})^{s+1} \tag{4}$$

$$\tilde{X}_s - \sqrt{3}\tilde{T}_s = (2-\sqrt{3})^{s+1}, s = 0,1,2,...... \tag{5}$$

(4) and (5) are solved, giving us

$$\tilde{X}_s = \frac{1}{2} f_s \quad \text{and} \quad \tilde{T}_s = \frac{1}{2\sqrt{3}} g_s$$

Where $f_s = (2+\sqrt{3})^{s+1} + (2-\sqrt{3})^{s+1}$

$$g_s = (2+\sqrt{3})^{s+1} - (2-\sqrt{3})^{s+1}$$

The series of unique non-zero integer solutions of (3) are derived as follows by using Brahmagupta's lemma between the solutions.

$$T_{s+1} = T_0 \tilde{X}_s + X_0 \tilde{T}_s$$
$$= \frac{1}{2} f_s + \frac{1}{\sqrt{3}} g_s$$
$$X_{s+1} = X_0 \tilde{X}_s + 3T_0 \tilde{T}_s$$
$$= f_s + \frac{\sqrt{3}}{2} g_s$$

By replacing the abovementioned values in (2),

$$x_{s+1} = X_{s+1} + T_{s+1}$$
$$= \frac{3}{2} f_s + \frac{5}{2\sqrt{3}} g_s \qquad (6)$$
$$y_{s+1} = X_{s+1} + 3T_{s+1}$$
$$= \frac{5}{2} f_s + \frac{9}{2\sqrt{3}} g_s \qquad (7)$$

Similarly, one may found

$$x_{s+2} = \frac{11}{2} f_s + \frac{19}{2\sqrt{3}} g_s$$
$$x_{s+3} = \frac{41}{2} f_s + \frac{71}{2\sqrt{3}} g_s$$
$$y_{s+2} = \frac{19}{2} f_s + \frac{33}{2\sqrt{3}} g_s \text{ and}$$
$$y_{s+3} = \frac{71}{2} f_s + \frac{123}{2\sqrt{3}} g_s$$

Additionally, their recurring links are discovered to be

$$x_{s+3} - 4x_{s+2} + x_{s+1} = 0$$
$$y_{s+3} - 4y_{s+2} + y_{s+1} = 0, s = -1,0,1,2......$$

Below are a few numerical examples that satisfy (1).

Table 1. Numerical Examples

s	x_{s+1}	y_{s+1}
-1	3	5
0	11	19
1	41	71
2	153	165
3	571	989
4	2011	3475
5	7953	13775
6	29681	51409
7	110771	191861

3.1 Below are a few intriguing recurring links amongst the solutions:

1. x_{s+1} *and* y_{s+1}, both are odd
2. $4y_{s+1} - 2y_{s+2} + 6x_{s+1} = 0$
3. $-x_{s+3} + 2x_{s+2} + y_{s+2} = 0$
4. $-x_{s+3} + x_{s+1} + 2y_{s+2} = 0$
5. $-2x_{s+2} + x_{s+1} + y_{s+2} = 0$
6. $4x_{s+1} - 2x_{s+2} + 2y_{s+1} = 0$
7. $y_{s+3} - 4y_{s+2} + y_{s+1} = 0$
8. $x_{s+3} - 4x_{s+2} + x_{s+1} = 0$
9. $y_{s+2} + 3x_{s+3} - 2y_{s+3} = 0$

4. Properties

4.1 One can create integer solutions for those hyperbolas by applying linear combinations among the solutions of (1).

From the equations (6) and (7), one may get

$$5x_{s+1} - 3y_{s+1} = -\frac{1}{\sqrt{3}} g_s$$
$$9x_{s+1} - 5y_{s+1} = f_s$$

By suitable search of $x = f_s$ and $y = -\frac{1}{\sqrt{3}} g_s$, one may get

$${g_s}^2 = 3y^2 \text{ and } {f_s}^2 = x^2$$

Here, one may found that

$${f_s}^2 + {g_s}^2 = 4$$
$$\Rightarrow x^2 - 3y^2 = 4$$

Where $x = 9x_{s+1}$ - $5y_{s+1}$ and $y = 5x_{s+1} - 3y_{s+1}$

Other options of solutions for different hyperbolas are obtained and exhibited in Table 2 as below.

Table 2. Hyperbolas

S. No	x, y	Hyperbola
1.	$x = 9x_{s+1} - 5y_{s+1}$ $y = 5x_{s+1} - 3y_{s+1}$	$x^2 - 3y^2 = 2^2$
	$x = 33x_{s+2} - 19y_{s+2}$ $y = 19x_{s+2} - 11y_{s+2}$	
	$x = 123x_{s+3} - 71y_{s+3}$ $y = 71x_{s+3} - 41y_{s+2}$	

2.	$x=33x_{s+1}-5y_{s+2}$ $y=19x_{s+1}-3y_{s+2}$	$x^2-3y^2=4^2$
	$x=9x_{s+2}-19y_{s+1}$ $y=5x_{s+2}-11y_{s+1}$	
	$x=123x_{s+2}-19y_{s+3}$ $y=5x_{s+2}-11y_{s+1}$	
	$x=33x_{s+3}-71y_{s+2}$ $y=19x_{s+3}-41y_{s+2}$	
3.	$x=123x_{s+1}-5y_{s+3}$ $y=51x_{s+1}-3y_{s+3}$	$x^2-3y^2=14^2$
	$x=9x_{s+3}-71y_{s+1}$ $y=5x_{s+3}-41y_{s+1}$	

4.2 One can generate answers for various parabolas by taking into account the linear combination of the solutions to (1).

From the equations (6) and (7), one may get

$$5x_{s+1}-3y_{s+1}=-\frac{1}{\sqrt{3}}g_s$$
$$9x_{s+1}-5y_{s+1}=f_s$$

By suitable search of $x=f_s{}^2$ and $y=-\frac{1}{\sqrt{3}}g_s$, one may get

$g_s{}^2=3y^2$ and $f_s{}^2=x$

Here, one may found that

$$f_s{}^2-g_s{}^2=4$$
$$\Rightarrow x^2-3y^2=4$$

where $x=9x_{2s+2}-5y_{2s+2}+2$ and $y=5x_{s+1}-3y_{s+1}$

Table 3 provides few other cases of parabolas

Table 3. Parabolas

S. No	x, y	**Parabola**
1.	$x=9x_{2s+2}-5y_{2s+2}+2$ $y=5x_{s+1}-3y_{s+1}$	$x-3y^2=2^2$
	$x=33x_{2s+3}-19y_{2s+3}+2$ $y=19x_{s+2}-11y_{s+2}$	
	$x=123x_{2s+4}-71y_{2s+4}+2$ $y=71x_{s+3}-41y_{s+2}$	

2.	$x=33x_{2s+2}-5y_{2s+3}+4$ $y=19x_{s+1}-3y_{s+2}$	$2x-3y^2=4^2$
	$x=9x_{2s+3}-19y_{2s+2}+4$ $y=5x_{s+2}-11y_{s+1}$	
	$x=123x_{2s+3}-19y_{2s+4}+4$ $y=5x_{s+2}-11y_{s+1}$	
	$x=33x_{2s+4}-71y_{2s+3}+4$ $y=19x_{s+3}-41y_{s+2}$	
3.	$x=123x_{2s+2}-5y_{2s+4}+14$ $y=51x_{s+1}-3y_{s+3}$	$7x-3y^2=14^2$
	$x=9x_{2s+4}-71y_{2s+2}+14$ $y=5x_{s+3}-41y_{s+1}$	

5. Conclusion and Future Scope

In this communication, we have attempted to identify all non-trivial solutions to the non-homogeneous binary quadratic equation.

To summarise, one can look for more options for solving the binary quadratic problem under consideration as well as quadratic equations with many variables.

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