A New Generalization of Pranav Distribution Using Weighting Technique

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Abstract-In this paper, we introduce a new generalization of Pranav distribution called as Weighted Pranav Distribution (WPD). The statistical properties of this distribution are derived and the model parameters are estimated by maximum likelihood estimation. Finally, an application to real data set is presented to examine the model performance.

Keywords: Pranav Distribution, Weighting Technique, Structural Properties and Maximum Likelihood Estimation.

I. INTRODUCTION

Fisher (1934) introduced the concept of weighted distributions and later it was introduced and formulated in general terms by Rao (1965), in connection with modeling statistical data where the usual practice of using standard distributions for the purpose was not found to be appropriate. Weighted distributions provide an approach to deal with model specification and data interpretation problems. In Rao’s paper (1965), he identified various situations that can be modeled by weighted distributions. These situations refer to instances where the recorded observations cannot be considered as a random sample from the original distributions. This may occur due to non observability of some events or damage caused to the original observation resulting in a reduced value or adoption of a sampling procedure which gives unequal chances to the units in the original. These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non experimental, non replicated and non random categories. The weighted distribution reduces to length biased distribution when the weight function considers only the length of the units. Different authors have reviewed and studied the various weighted probability models and illustrated their applications in different applied fields. Weighted distributions were applied in various research areas related to reliability, biomedicine, ecology and branching processes. For survival data analysis, Jing (2010) introduced the weighted inverse Weibull distribution and beta-inverse Weibull distribution as a new lifetime models. Ayesh, (2017) discussed the Size Biased Lindley Distribution as a new life time distribution and discussed its various statistical properties. Shankar (2017) discussed a Size-Biased Poisson-Shanker distribution and its applications to handle various count data sets. Para and Jan (2018) introduced the Weighted Pareto type II Distribution as a new model for handling medical science data and studied its statistical properties and applications. Recently Hassan et al. (2018) introduced two weighted probability models with applications in handling various lifetime data from different applied fields.

In the present paper, we use the weighting technique and propose a two parameter probability distribution which is the new generalization of Pranav distribution given by Rama Shanker (2018). This new model provides a better fit to data regarding successive March precipitation (in inches) studied by Hinkley (1977).

II. WEIGHTED PRANAV DISTRIBUTION (WPD)

Pranav distribution is a newly proposed lifetime model formulated by Rama Shanker (2018) for several engineering applications and calculated its various characteristics including stochastic ordering, moments, order statistics, Renyi entropy, Stress-Strength reliability and ML estimation. Probability density function (pdf) of Pranav Distribution (AD) is given by
The probability density function of the weighted Pranav model is given by:

\[ f(x; \theta) = \frac{\theta^4}{\theta^4 + 6} (\theta + x^3)e^{-\theta x}; \quad x > 0, \theta > 0 \]  \hspace{1cm} (2.1)

The corresponding cdf of (2.1) is given by

\[ F(x; \theta) = 1 - \left[ 1 + \frac{\theta x (\theta^2 + 3\theta x + 6)}{\theta^4 + 6} \right] e^{-\theta x}; \quad x > 0, \theta > 0 \]  \hspace{1cm} (2.2)

Assume \( X \) is a non-negative random variable with probability density function (pdf) \( f(x) \). Let \( W(x) \) be the weight function which is a non-negative function, then the probability density function of the weighted random variable \( X_w \) is given by:

\[ f_w(x) = \frac{W(x)f(x)}{E(w(x))}, \quad x > 0, \]

where \( w(x) \) be a non-negative weight function and \( E(w(x)) = \int w(x)f(x)dx < \infty \).

In this paper, we have considered the weight function as \( w(x) = x^c \) to obtain the weighted Pranav model. The probability density of weighted Pranav distribution is given as:

\[ f_w(x, c, \theta) = \frac{x^c f(x, \theta)}{E(x^c)}, \]

\[ f_w(x; c, \theta) = \frac{x^c \theta^{(c+4)}(\theta + x^3)e^{-\theta x}}{c!(\theta^4 + (c+1)(c+2)(c+3))}, \quad x > 0, \theta > 0, c > 0, \]  \hspace{1cm} (2.3)

Where \( E(x^c) = \frac{c!(\theta^4 + (c+1)(c+2)(c+3))}{\theta^c(\theta^4 + 6)} \)

The corresponding cdf of weighted Pranav Distribution (WPD) is obtained as

\[ F_w(x; c, \theta) = \int_0^x f_w(x; c, \theta)dx \]

\[ = \int_0^x \frac{x^c \theta^{(c+4)}(\theta + x^3)e^{-\theta x}}{c!(\theta^4 + (c+1)(c+2)(c+3))} dx, \quad \text{put} \quad \theta x = t, \quad \theta dx = dt, \]

as \( x \to 0, t \to 0 \) and \( x \to x, t \to \theta x \), after simplification

\[ F_w(x; c, \theta) = \frac{\theta^4}{c!(\theta^4 + (c+1)(c+2)(c+3))} \left( \gamma(c+1, \theta x) + \frac{1}{\theta^4} \gamma(c+4, \theta x) \right), \]  \hspace{1cm} (2.4)

\[ x > 0, \theta > 0, c > 0, \]

where \( \theta \) and \( c \) are positive parameters and \( \gamma(s, x) = \int_0^x t^{s-1}e^{-t}dt \) is a lower incomplete gamma function.

The graphs of probability density function and cumulative distribution function are plotted for different values of parameters \( \theta \) and \( c \) given in fig.1 and fig. 2 respectively. It illustrates that the density function of weighted Pranav distribution is positively skewed. For fixed \( \theta \) it becomes more and more flatter as the value of \( c \) is increases. Fig. 2 shows the graph of
distribution function which is an increasing function, as the weighted parameter $c$ increases initial rise of cdf curve also decreases.

### III. SPECIAL CASES

Case 1: If we put $c = 0$, then weighted Pranav distribution (2.1) reduces to Pranav distribution with probability density function as:

$$f(x; \theta) = \frac{\theta^4}{\theta^4 + 6} (\theta + x^3)e^{-\theta x}, \quad x > 0, \theta > 0,$$

### IV. RELIABILITY ANALYSIS

In this section, we have obtained the reliability, hazard rate, reverse hazard rate of the proposed weighted Pranav distribution.

#### A. Reliability function $R(x)$

The reliability function is defined as the probability that a system survives beyond a specified time. It is also referred to as survival or survivor function of the distribution. It can be computed as complement of the cumulative distribution function of the model. The reliability function or the survival function of weighted Pranav distribution is calculated as:

$$R_w(x, c, \theta) = 1 - \frac{\theta^4}{c!(\theta^4 + (c + 1)(c + 2)(c + 3))} \left( \gamma(c + 1, \theta c) + \frac{1}{\theta^4} \gamma(c + 4, \theta c) \right)$$

$$x > 0, \theta > 0, c > 0,$$

#### B. Hazard Function

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality and is given as:

$$H.R = h(x; c, \theta) = \frac{f_w(x, \theta)}{R_w(x, \theta)},$$

$$= \frac{x^c \theta^{c+4} (\theta + x^3)e^{-\theta x}}{c!(\theta^4 + (c + 1)(c + 2)(c + 3) - \theta^4 \gamma(c + 1, \theta c) + \frac{1}{\theta^4} \gamma(c + 4, \theta c))}$$

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C. Reverse Hazard Rate
The reverse hazard rate of the weighted Pranav distribution are respectively given as:

\[ R.H.R = h_r(x, c, \theta) = \frac{f_w(x, \theta)}{F_w(x, \theta)} = \frac{x^r \theta^{c+r+4} (\theta + x^3) e^{-\theta x}}{\theta^4 (\gamma (c+1, \theta x) + \frac{1}{\theta^4 \gamma (c+4, \theta x))}}, \quad x > 0, \theta > 0, c > 0, \]

V. STATISTICAL PROPERTIES
In this section, the different structural properties of the proposed weighted Pranav model have been evaluated. These include moments, harmonic mean, moment generating function and characteristic function.

A. Moments: Suppose X is a random variable following weighted Pranav distribution with parameter \( \theta \), and then the rth moment for a given probability distribution is given by

\[ \mu_r = E\left(X^r\right) = \int_0^\infty x^r f_w(x, \theta) dx \]

\[ = \int_0^\infty x^r \frac{x^c \theta^{c+4} (\theta + x^3) e^{-\theta x}}{c! (\theta^4 + (c+1)(c+2)(c+3))} dx \]

\[ \mu_r = \frac{(c+r)! (\theta^4 + (c+r+1)(c+r+2)(c+r+3))}{\theta^r c! (\theta^4 + (c+1)(c+2)(c+3))} \]

\[ \mu_1 = \frac{(c+1)(\theta^4 + (c+1)(c+2)(c+3))}{(\theta^4 + (c+1)(c+2)(c+3))} \]

\[ V(X) = \frac{(c+1)(c+2)(\theta^2 + (c+3)(c+4) + 2\theta(c+3))}{\theta^2 (\theta^2 + 2\theta(c+1) + (c+1)(c+2))} \]

\[ V(X) = \left[ (c+1)(\theta^2 + (c+2)(c+3) + 2\theta(c+2)) \right]^2 \]

B. Harmonic mean
The harmonic mean for the proposed model is computed as:

\[ H.M = E\left(\frac{1}{X}\right) = \int_0^\infty \frac{1}{x} f_w(x, c, \theta) dx \]

\[ = \int_0^\infty \frac{1}{x} \frac{x^c \theta^{c+4} (\theta + x^3) e^{-\theta x}}{c! (\theta^4 + (c+1)(c+2)(c+3))} dx \]

\[ H.M = \frac{c(\theta^4 + (c+1)(c+2)(c+3))}{\theta(\theta^4 + c(c+1)(c+2))}, \quad \theta > 0, c > 0, \]

C. Moment generating function and Characteristic function of Weighted Pranav Distribution (WAD)
We will derive moment generating function and characteristic function of WPD in this section.

Theorem 1.1: If X has the WPD \( (c, \theta) \), then the moment generating function \( M_X(t) \) and the characteristic function \( \psi_X(t) \) has the following form

\[ M_X(t) = \frac{\theta^{c+4}}{\theta^4 + (c+1)(c+2)(c+3)} \left[ \frac{\theta(\theta - t)^3 + (c+1)(c+2)(c+3)}{(\theta - t)^{c+4}} \right] \]

and
\[ \psi_x(t) = \frac{\Theta^{c+4}}{\Theta^4 + (c+1)(c+2)(c+3)} \left\{ \frac{\Theta(t - \theta)^3 + (c+1)(c+2)(c+3)}{(\theta - t)^{c+4}} \right\} \text{respectively.} \]

**Proof:** We begin with the well known definition of the moment generating function given by

\[ M_X(t) = E(e^{tx}) = \int e^{tx} f(x; \theta, c) \, dx \]

\[ = \frac{\Theta^{c+4}}{c!(\theta^4 + (c+1)(c+2)(c+3))} \int_0^\infty [x^c(\theta + x^3)e^{-(\theta - t)x}] \, dx \]

\[ \Rightarrow M_X(t) = \frac{\Theta^{c+4}}{(\Theta^4 + (c+1)(c+2)(c+3))} \left\{ \frac{\Theta(t - \theta)^3 + (c+1)(c+2)(c+3)}{(\theta - t)^{c+4}} \right\} \]

Also we know that \[ \psi_x(t) = M_X(it) \]

Therefore,

\[ \psi_x(t) = \frac{\Theta^{c+4}}{(\Theta^4 + (c+1)(c+2)(c+3))} \left\{ \frac{\Theta(t - \theta)^3 + (c+1)(c+2)(c+3)}{(\theta - t)^{c+4}} \right\} \]

**VI. ORDER STATISTICS**

Let \( X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)} \) be the ordered statistics of the random sample \( X_1, X_2, X_3, \ldots, X_n \) drawn from the continuous distribution with cumulative distribution function \( F_X(x) \) and probability density function \( f_X(x) \), then the probability density function of rth order statistics \( X_{(r)} \) is given by:

\[ f_{(r)}(x, c, \theta) = \frac{n!}{(r-1)!(n-r)!} f(x) \left[ F(x) \right]^{r-1} \left[ 1 - F(x) \right]^{n-r} \quad r = 1, 2, 3, \ldots, n \]

Using the equations (2.3) and (2.4), the probability density function of rth order statistics of weighted Pranav distribution is given by:

\[ f_{(r)}(x, c, \theta) = \frac{n!}{(r-1)!(n-r)!} \frac{x^c \Theta^{c+4}(\theta + x^3)e^{-\theta x}}{c!(\theta^4 + (c+1)(c+2)(c+3))} \left[ \frac{\Theta^4}{c!(\theta^4 + (c+1)(c+2)(c+3))} \left( \gamma(c+1, \theta x) + \frac{1}{\theta^4} \gamma(c+4, \theta x) \right) \right]^{r-1} \left[ 1 - \frac{\Theta^4}{c!(\theta^4 + (c+1)(c+2)(c+3))} \left( \gamma(c+1, \theta x) + \frac{1}{\theta^4} \gamma(c+4, \theta x) \right) \right]^{n-r} \]

Then, the pdf of first order \( X_{(1)} \) weighted Pranav distribution is given by:

\[ f_{(1)}(x, c, \theta) = n \frac{x^c \Theta^{c+4}(\theta + x^3)e^{-\theta x}}{c!(\theta^4 + (c+1)(c+2)(c+3))} \left[ 1 - \frac{\Theta^4}{c!(\theta^4 + (c+1)(c+2)(c+3))} \left( \gamma(c+1, \theta x) + \frac{1}{\theta^4} \gamma(c+4, \theta x) \right) \right]^{n-1} \]

and the pdf of nth order \( X_{(n)} \) weighted Pranav model is given as:

\[ f_{(n)}(x, c, \theta) = n \frac{x^c \Theta^{c+4}(\theta + x^3)e^{-\theta x}}{c!(\theta^4 + (c+1)(c+2)(c+3))} \left[ \frac{\Theta^4}{c!(\theta^4 + (c+1)(c+2)(c+3))} \left( \gamma(c+1, \theta x) + \frac{1}{\theta^4} \gamma(c+4, \theta x) \right) \right]^{n-1} \]

**VII. METHOD OF MAXIMUM LIKELIHOOD ESTIMATION OF WEIGHTED PRANAV DISTRIBUTION**

This is one of the most useful method for estimating the different parameters of the distribution. Let \( X_1, X_2, X_3, \ldots, X_n \) be the random sample of size n drawn from weighted Pranav distribution, then the likelihood function of weighted Pranav distribution is given as:
and equating the result to zero, we obtain the following normal equations,

\[ \begin{align*}
\sum_{i=1}^{n} x_i (\theta^c + 4c) e^{-\theta x} &= 0 \\
\sum_{i=1}^{n} x_i (\theta^{c+4} + (c+1)(c+2)(c+3)) &= 0
\end{align*} \]

The log likelihood function becomes:

\[ \log L = c \log \sum_{i=1}^{n} x_i + n(c + 4) \log \theta + \log \left( c! \theta^c + (c + 1)(c + 2)(c + 3) \right) \]

Differentiating the log-likelihood function with respect to \( \theta \) and \( c \). This is done by partially differentiate (7.1) with respect to \( \theta \) and \( c \) and equating the result to zero, we obtain the following normal equations,

\[ \begin{align*}
\frac{d \log L}{d \theta} &= \frac{n(c + 4)}{\theta} - \sum_{i=1}^{n} x_i \theta^c - (c + 1)(c + 2)(c + 3) = 0 \\
\frac{d \log L}{dc} &= \log \left( \sum_{i=1}^{n} x_i \right) + n \log \theta + \frac{n(3c^2 + 12c + 11)}{c!} - n \log (\theta^c + (c + 1)(c + 2)(c + 3)) = 0
\end{align*} \]

By solving equations (7.2) and (7.3), the maximum likelihood estimators of the parameters of the weighted Pranav distribution are obtained using the numerical methods like Newton Raphson method.

We can compute the maximized unrestricted and restricted log likelihoods to construct the likelihood ratio (LR) statistics for testing the significance of weighted parameter of the proposed model. For example, we can use LR test to check whether the fitted weighted Pranav distribution for a given data set is statistically “superior” to the fitted Pranav distribution. In any case, hypothesis tests of the type \( H_0: \theta = \theta_0 \) versus \( H_1: \theta \neq \theta_0 \) can be performed using LR statistics. In this case, the LR statistic for testing \( H_0 \) versus \( H_1 \) is \( \omega = 2(L(\hat{\theta}) - L(\hat{\theta}_0)) \) where \( \hat{\theta} \) and \( \hat{\theta}_0 \) are the MLEs under \( H_1 \) and \( H_0 \). The statistic \( \omega \) is asymptotically (as \( n \to \infty \)) distributed as \( \chi^2_k \), with \( k \) degrees of freedom which is equal to the difference in dimensionality of \( \hat{\theta} \) and \( \hat{\theta}_0 \). \( H_0 \) will be rejected if the LR-test p-value is <0.05 at 95% confidence level.

### VIII. APPLICATIONS OF WEIGHTED PRANAV DISTRIBUTION

In this section usefulness of the weighted Pranav distribution is illustrated by using the real-life data example. The data consist of thirty successive March precipitation (in inches) observations studied by Hinkley (1977). The data set is provided below in table 1. Analysis of the data set is performed using R software.

**Table 1: Data regarding successive March precipitation (in inches) studied by Hinkley (1977).**

<table>
<thead>
<tr>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
<th>Value 5</th>
<th>Value 6</th>
<th>Value 7</th>
<th>Value 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.77</td>
<td>1.74</td>
<td>0.81</td>
<td>1.20</td>
<td>1.95</td>
<td>1.20</td>
<td>0.47</td>
<td>1.43</td>
</tr>
<tr>
<td>3.00</td>
<td>3.09</td>
<td>1.51</td>
<td>2.10</td>
<td>0.52</td>
<td>1.62</td>
<td>1.31</td>
<td>0.32</td>
</tr>
<tr>
<td>2.81</td>
<td>1.87</td>
<td>1.18</td>
<td>1.35</td>
<td>4.75</td>
<td>2.48</td>
<td>0.96</td>
<td>1.89</td>
</tr>
<tr>
<td>2.05</td>
<td>0.90</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In order to compare the weighted Pranav distribution and Pranav distribution, we are using the AIC (Akaike information criterion) given by Akaike (1976), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion) given by Schwarz (1987). The better distribution corresponds to lesser AIC, AICC and BIC values.

\[
\text{AIC} = 2k - 2\log L \\
\text{AICC} = \text{AIC} + \frac{2k(k + 1)}{n - k - 1} \\
\text{BIC} = k \log n - 2\log L
\]

where \( k \) is the number of parameters in the statistical model, \( n \) is the sample size and \( -\log L \) is the maximized value of the log-likelihood function under the considered model. Model parameters are estimated using maximum likelihood method. From table 2, it has been observed that weighted Pranav distribution have the lesser AIC, AICC, \( -\log L \) and BIC values as compared to Pranav distribution. Hence we can conclude that the weighted Pranav distribution leads to a better fit than the Pranav
distribution. We also test the significance of weighted parameter $c$ with the help of Likelihood ratio test. Weighted parameter $c$ plays a significant role in analyzing the data regarding successive March precipitation (in inches) studied by Hinkley (1977). Figure 3 provides a graphical overview of the fitted models to the data set given in table 1 and it is clear that weighted Pranav distribution is better model as comparison to Pranav distribution for analyzing the data regarding successive March precipitation (in inches) studied by Hinkley (1977).

### Table 2: Parameter Estimates, AIC, AICC, BIC, -logL Criterion and Likelihood Ratio statistic values for precipitation data set reported by Hinkley (1977).

<table>
<thead>
<tr>
<th>Probability Model</th>
<th>Parameter Estimates</th>
<th>$-\log L$</th>
<th>AIC</th>
<th>AICC</th>
<th>BIC</th>
<th>Likelihood Ratio Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weighted Pranav</td>
<td>$\hat{c} = 2.17, \hat{\theta} = 2.83$ (0.79, 0.48)</td>
<td>38.211</td>
<td>80.423</td>
<td>80.867</td>
<td>83.225</td>
<td></td>
</tr>
<tr>
<td>Distribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>14.56</td>
</tr>
<tr>
<td>Pranav Distribution</td>
<td>$\hat{c} = 1.51 (0.11)$</td>
<td>45.490</td>
<td>92.981</td>
<td>93.124</td>
<td>94.382</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3: Model Fitting to the Precipitation (in inches) data, Pranav vz Weighted Pranav distribution

![Figure 3: Model Fitting to the Precipitation (in inches) data, Pranav vz Weighted Pranav distribution](image)

### IX. CONCLUSION

In the present study, we have introduced a new generalization of the Pranav distribution called as weighted Pranav distribution. The subject distribution is generated by using the weighting technique and taking the one parameter Pranav distribution as the base distribution. Some mathematical properties along with the application part are discussed in detail.
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