Strong Regularity in Near-Rings

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Abstract — The aim of this paper is to prove some concepts related with strong regularity and strong reducedness in near-rings. We arrive at a conclusion that all reduced and regular near-rings are strongly regular and strongly reduced near-rings. Also we discuss the idea of semicentral in near-rings.

Keywords — Near-rings, Regular, Reduced, Strongly regular, Strongly reduced, Right semicentral

I. INTRODUCTION

Throughout this paper we work with right near-rings. Mason[1] introduced the notion of strong regularity of near-rings and characterized left regular zero-symmetric unital near-rings. Reddy and Murty [2] extended the results in [1] to arbitrary near-rings and proved that the concepts of left regularity, left strong regularity and right regularity in near-rings are equivalent and these imply right strong regularity. Yong Uk Cho and Yasuyuki Hirano[3] showed that the strong regularity in near-ring is equivalent property (* ) in[2]. Narmada and Anil Kumar[4] characterize the strong regularity of near-rings.

We will use the following notations:

Given a near-ring \( N \) , \( N_0 = \{ n \in N : n0 = 0 \} \) which is called the zero-symmetric part of \( N \), \( N_0 = \{ n \in N : n0 = n \} = \{ n \in N : mn' = n \text{ for every } n' \in N \} \) which is called constant part of \( N \). Clearly, \( N_0 \) and \( N_c \) are subnear-rings of \( N \). A near-ring \( N \) is called zero-symmetric if \( N = N_0 \) and \( N \) is called a constant near-ring if \( N = N_c \). For basic concepts and notations we shall refer to Pilz[5].

II. PRELIMINARIES

The concept of homomorphism on near-rings are like on rings. \( Z_n \) is an abelian group under addition modulo \( n \). Let \( N, N' \) be near-rings, \( h : N \to N' \) is called near-ring homomorphism if for every \( m, n \in N \) such that \( h(m + n) = h(m) + h(n) \) and \( h(mn) = h(m)h(n) \). A near-ring \( N \) is called left strongly regular if for every \( a \in N \), there exist \( x \in N \) such that \( a = xa^2 \) and left regular if for every \( a \in N \), there exist \( x \in N \) such that \( a = xa^2 \) and \( a = axa \). Right strong regularity and right regularity can be defined in symmetric way. \( N \) is strongly regular if it is both left and right strongly regular. We can say that \( N \) is reduced if \( N \) has no nonzero nilpotent elements, that is, for each \( a \in N \), \( a^n = 0 \), for some positive integer \( n \) implies \( a = 0 \). In ring theory Mc Coy proved that \( N \) is reduced if and only if for each \( a \in N \), \( a^2 = 0 \) implies \( a = 0 \). A near-ring \( N \) is said to be strongly reduced if for each \( a \in N \), \( a^2 \in N_c \) implies \( a \in N_c \).

An idempotent \( e \in N \) is right semicentral if \( en = ne \) for each \( n \in N \). Similarly, left semicentral can be defined in a symmetric way. A near-ring in which every idempotent is right semicentral is called right semicentral near-ring. An idempotent \( e \) in \( N \) is central if \( en = ne \) for each \( x \in N \).

III. CHARACTERIZATION OF HOMOMORPHISM ON \( Z_n \) ([6]):

Suppose \( f : Z_m \to Z_n \) is a group homomorphism and assume \( f(1) = k \), then for \( m \in N \), \( f(m) = mk \) and \( f(-m) = -mk \). Thus \( x \in Z, f(x) = xf(1) = xk \)
\( f(0) = 0f(1) \). Then for \( x \in Z_m, f(x) = xk \) for some \( k \in Z_n \).

That is \( f : Z_m \to Z_n \) is a group homomorphism and \( f(1) = k \), then the homomorphism has the form \( f(x) = xk \) (mod \( n \)).

Conversely, if \( k \) is a solution of \( mx \equiv 0 \) (mod \( n \)) then \( f(x) = xk \) (mod \( n \)) is a homomorphism from \( Z_m \to Z_n \).

Theorem 3.1 : The function \( f : Z_m \to Z_n \) given by \( f(x) = xk \) for some \( k \in Z_n \), fixed is homomorphism of groups if and only if \( mk \equiv 0 \) (mod \( n \)).
Corollary 3.2 : The function $f: Z_m \rightarrow Z_n$ is a homomorphism and $f(x) = xk$ where $k$ is the solution the system $mk \equiv 0 (mod \ n)$ and $(m,n) = 1$ , is an on to homomorphism.

Proposition 3.3 : Suppose $f: Z_m \rightarrow Z_n$ is a ring homomorphism and assume $f(1) = k$, since every ring homomorphism is a group homomorphism, $f(x) = xk (mod \ n)$ is a ring homomorphism if and only if $k$ is the solution of the system $mk \equiv 0 (mod \ n)$. Also $k = f(1) = f(1.1) = [f(1)]^2 = k^2 (k \text{ is idempotent})$. That is $k$ is also a solution of the system $x^2 = x (mod \ n)$.

IV. STRONGLY REGULAR NEAR-RINGS

G Pilz[5], gives the description of near-rings of low order and from the above argument of homomorphism on near-rings , [6] gives the idea to construct the near-rings of low order on $Z_n , n \leq 7$. We know that every strongly regular near-ring is strongly reduced (Proposition 1 of [3]).

Lemma 4.1 : [3] Let $N$ be a strongly reduced near-ring. If for any $b, b \in N$ with $ab \in N_c , then ba \in N_c .$ and for every $x \in N , axa , bxa \in N_c$ . Furthermore, $b^n \in N_c$ implies $ab \in N_c , for$ each positive integer $n$.

Theorem 4.2 : If $N$ is strongly regular , then it is strongly reduced and regular.

Proof: By Proposition 1 of [3] , $N$ is strongly reduced. Let $a \in N$. Then, there exists $x \in N$ such that $a = xa^2$ . Since $(a - axa)a = 0 \in N_c$, by lemma 4.1, $a(a - axa), axa(a - axa) \in N_c$ . Thus $(a - axa)^2 \in N_c$ and this implies $a - axa \in N_c$ . Hence, $0 = (a - axa)a = a - axa$ showing that $= axa$ . Thus, $N$ is regular.

Theorem 4.3 : A near-ring $N$ is strongly regular if and only if it is right semicentral and regular.

Proof. Assume that $N$ is strongly regular . Then $N$ is strongly reduced and regular, by Theorem 4.2. Let $e^2 = e$. Now, $(e - en)e = 0 \in N_c$ . Hence, $en(en - en), en(e - en) \in N_c$ by Lemma 4.1. So, $(en - en)^2 \in N_c$ and this implies $en - en \in N_c$ . Hence, $0 = (en - en)e = en - en$, showing that $en = en$. Thus $N$ is right semicentral.

Conversely, assume that $N$ is right semicentral and regular. Let $a \in N$. Since $N$ is regular,a = axa , for some $x \in N$ . Then $xa$ is idempotent. Now, $a = (ax)a = (a(ax)x(ax))a = (ax)x^2a^2 = ax^2a^2 = yo^2$, where $y = ax^2 \in N$.

Theorem 4.4 [4] : Let $N$ be a near-ring. Then $N$ is strongly regular if and only if it is strongly reduced and regular.

From Clay’s [7] table [6] classifies reduced and regular near-rings of order $\leq 7$ , which are strongly regular and strongly reduced near-rings.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Zero-symmetric, reduced and regular</th>
<th>Nonzero-symmetric, reduced and regular</th>
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<tr>
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<td>$Z_7$</td>
<td>18, 20, 21, 22, 23, 24</td>
<td>19</td>
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</tbody>
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REFERENCES