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A Study of Mathematical Model for Pollutant Concentration and Dissolved Oxygen Concentration in the Kshipra River (India)

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Abstract— This paper addresses a mathematical model for pollutant and dissolved oxygen concentrations in a river. The physical and mass conservation laws governing the interaction of pollutant and dissolved oxygen are expressed as non-linear one-dimensional, time-dependent, coupled partial differential equations and are solved numerically, for zero-diffusion case, using fourth-order Runge-Kutta method. Time required for the pollutant to decay completely and dissolved oxygen to reach its saturation level is obtained graphically through simulation study. A particular case of the River Kshipra (Madhya Pradesh, India) is studied in detail. The result indicates the self-purification ability of the river at four different monitoring stations along its stretch in Ujjain city, using some standard parameters of water quality. The study is motivated by the pathetic condition of the river, as it receives large amount of untreated sewage daily. This study provides a better understanding of the complex phenomena of pollution transportation in the River Kshipra.

Keywords— *Dissolved Oxygen, Biochemical Oxygen Demand, Diffusion, Advection, Runge-Kutta Method.*

I. INTRODUCTION

Water is a vital resource that is essential for all living beings. The problem of water pollution has become a common environmental problem nowadays. In recent years, most of the Indian rivers viz., Ganga, Yamuna, Narmada, Godavari etc. are reported to be severely polluted and various attempts are done by the government of India to check, predict and control their water quality. The advancement in mathematical modeling has led to the development of water quality models, which enable us to handle the problem of pollution without directly interfering with it. Mathematical models for water quality problems provide the ability to predict the pollutant consideration levels of a river. It is a very powerful approach that greatly enhances the decision support tools used for water resource management.

Many factors need to be considered while assessing the quality of water, but since the ecology of a river largely depends on the Dissolved Oxygen (DO) concentration, it seems to be a convenient criterion for measuring the degree of pollution of the river which is mainly polluted by organic pollutants and sewage.

In this study, the mathematical model devised by Pimpunchat et al. [1] is studied in detail. The results are obtained by fourth order Runge-Kutta method for zero diffusion case in the absence of flow. The same is applied to the holy river Kshipra by collecting the required data for four monitoring stations along its stretch.

This research provides a tool for policy development and effective management of water quality issues for the river. This will aid decisions concerning future restrictions to be imposed on sewage discharge, maintaining desired DO levels and controlling other urban practices.

Rest of the paper is organized as follows, Section I contains the introduction of the crucial water pollution problem in India, Section II contains the related work of the water pollution modeling field, Section III contain the details of the River Kshipra whose wretched state motivated this study, Section IV explains the all-inclusive mathematical model, Section V contains the numerical solution of the model for the zero-diffusion case along with the simulation study, Section VI describe results and their interpretation using graphical representations and Section VII concludes research work with future directions.

II. RELATED WORK

At first, Streeter and Phelps [2] developed a model equation to study the declination of DO concentration in 1925, which was solved analytically and represented in the form of DO sag curve.

A broad review of water quality models has been produced by Chapra [3]. Pimpunchat et al. [4] have presented a set of coupled partial differential equations describing river pollution and obtained some analytical solution for steady state cases. However, for non-steady complicated models analytical solutions cannot be derived. In those cases, approximated solutions can be obtained by using efficient numerical methods. For example, Hasadsri and Maleewong [5] have used finite element method, Simon and Koya [6] splitted the problem and then used a set of different numerical procedures to obtain approximated solutions. A comprehensive review of numerical methods, like finite difference, finite element and finite volume method has been produced by Benedini and Tsakiris [7]. Since then, various models were proposed [8, 9, 10] and were solved both analytically and numerically.

III. MOTIVATION : THE KSHIPRA RIVER

The holy Kshipra River is in Madhya Pradesh state of Central India. It originates from Kokri Bardi Hills of Vindhya Range and flows across the Malwa Plateau, mainly through the city Ujjain, and joins the Chambal River near Kalu-Kher village. It is one of the sacred rivers in Hinduism. Every 12 years, the Simhastha (Kumbh) fair takes place on riverside ghats. It is a perennial river, but has lost its nature and now it runs dry for a period of 5 to 6 months per year.

The total length of the river is about 195 km with 6123 sq.km catchment area [11]. This study is carried out particularly in Ujjain city. River enters the city at Triveni Ghat and leaves at Kaliyadeh Stop Dam, passing through other important ghats namely Gaughat, Ramghat and Siddhwat ghat. Drains carrying sewage and wastewater of the city join it at various locations. This stretch which is approximately 19.8 km in length is most critical from the pollution point of view and need to be addressed properly [12]. The condition gets worse during festive occasions when thousands of people use the flowing water of Kshipra for bathing purposes. In April-May, 2016, Kumbh Mela was held in which more than 60 million people dipped in the holy river. Due to these practices river is dying and these kinds of celebrations will increasingly seem hollow if intelligent efforts are not made to save the river.

IV. THE MATHEMATICAL MODEL

For Kshipra River, it is assumed that the pollutants are largely biological wastes which undergo various biochemical and biodegradation processes using DO. Here, a coupled equation for the pollutant and DO concentration is considered. The coupling occurs because the oxygen reacts with the pollutant producing harmless compounds. For simplicity, it is assumed that the flow is one-dimensional with uniform cross-sectional area.
For the model, the variables and parameters used are listed in Table 1.

Table 1. Variables and Parameters

VARIABLES		
t	Time (day)	
x	Position (m)	
$P(x,t)$	Pollutant Concentration (kg m^{-3})	
$X(x,t)$	Dissolved Oxygen Concentration (kg m^{-3})	
PARAMETERS		
L	Polluted length of the river (under study) (m)	19790
A	Cross-sectional area of river (average)(m^2)	1000
k	Half-saturated oxygen demand concentration for pollutant decay (kg m^{-3})	0.007**
α	Mass transfer of oxygen from air to water(m^2 day^{-1}); $\alpha =$ re-aeration rate*A' From Chapra, the re-aeration rate=0.055 day^{-1}, A'= estimated average width of 100 * unit length of 1)	5.5*
S	Saturated oxygen concentration (kg m^{-3})	0.01*
q	Added pollutant rate along the river (kg m^{-1} day^{-1})	0.6**
K_1	Degradation rate coefficient for pollutant (day^{-1}), based on molecular weights in the chemical reaction $K_1 = 3/16\, K_2$	9/16
K_2	De-aeration rate coefficient at for DO (day^{-1})	3**

*[3,4]
** estimated

The standard equations are mass balance equations. They describe the rate of change of the concentration of pollutant $P(x,t)$ and the concentration of DO $X(x,t)$, with position x (m) ($0 \le x \le L$) and time t (days) , and can be expressed in one-dimension as:

$$\frac{\partial\big(AP(x,t)\big)}{\partial t} = D_p \frac{\partial^2\big(AP(x,t)\big)}{\partial x^2} - \frac{\partial\big(vAP(x,t)\big)}{\partial x} - K_1 \frac{X}{X+k} AP(x,t) + qH(x) \quad (1)$$

$$\frac{\partial\big(AX(x,t)\big)}{\partial t} = D_x \frac{\partial^2(AX(x,t))}{\partial x^2} - \frac{\partial(vAX(x,t))}{\partial x} - K_2 \frac{X}{X+k} AP(x,t) + \alpha(S-X) \quad (2)$$

The physical interpretation of the terms used in (1) and (2) are as follows:

- $D_p \frac{\partial^2(AP(x,t))}{\partial x^2}$ represents diffusion of pollutant,
- $\frac{\partial(vAP(x,t))}{\partial x}$ represents advection (convection) of pollutant,
- $K_1 \frac{X}{X+k} AP(x,t)$ represents the removal of pollutant by DO,
- $qH(x)$ represent addition of pollutant along the x-direction, here $H(x)$ is the Heaviside function, i.e.

$$H(x) = \begin{cases} 1, & 0 < x < L \\ 0, & otherwise \end{cases} \qquad (3)$$

- $D_x \frac{\partial^2(AX(x,t))}{\partial x^2}$ represents diffusion of oxygen as it enters through the water surface from the environment,
- $\frac{\partial(vAX(x,t))}{\partial x}$ represents advection (convection) of DO,
- $K_2 \frac{X}{X+k} AP(x,t)$ represents oxygen consumed by the pollutants to oxidise themselves and produce harmless compounds, and
- $\alpha(S - X(x,t))$ represents the addition of oxygen through the surface at a rate proportional to the degree of saturation of dissolved oxygen $(S - X(x,t))$.

For the above model, here it is assumed that the advection along the river can be neglected i.e. $v = 0$, because water remains almost stagnant during non-monsoon period. So, the system of equations (1) and (2) simplifies to

$$\frac{\partial(AP(x,t))}{\partial t} = D_p \frac{\partial^2(AP(x,t))}{\partial x^2} - K_1 \frac{X}{X+k} AP(x,t) + qH(x) \qquad (4)$$

$$\frac{\partial(AX(x,t))}{\partial t} = D_x \frac{\partial^2(AX(x,t))}{\partial x^2} - K_2 \frac{X}{X+k} AP(x,t) + \alpha(S - X) \qquad (5)$$

This model is solved numerically for non-steady case. For this, let us restrict the independent variables to the regions $0 \le x \le L$ and $t \ge 0$. The boundary conditions are considered as $P(0,t) = P(L,t) = 0$ and $X(0,t) = X(L,t) = S$. The initial conditions are considered as $P(x,0) = 7S = 0.07$ and $X(x,0) = S = 0.01$. Also, the parametric values chosen are listed in the Table 1.

V. NUMERICAL SIMULATION FOR ZERO DIFFUSION

Considering the case when the dispersion can be taken to be zero; i.e. $D_p = 0,\ D_x = 0$. In this case the system of partial differential equations (4) and (5) becomes a system of ordinary differential equations as follows

$$\frac{d(AP(t))}{dt} = -K_1 \frac{X}{X+k} AP(t) + q \qquad (6)$$

$$\frac{d(AX(t))}{dt} = -K_2 \frac{X}{X+k} AP(t) + \alpha(S - X) \qquad (7)$$

Assuming the parameters A, q, α, K_1, K_2 and S to be constant,

$$\frac{dP(t)}{dt} = -K_1 \frac{X}{X+k} P(t) + \frac{q}{A} \qquad (8)$$

$$\frac{dX(t)}{dt} = -K_2 \frac{X}{X+k} P(t) + \alpha(S - X) \qquad (9)$$

Here, the effect of cross-sectional area from the last term of (9) can be omitted, since oxygen is added to the river only through water surface.

Thus only the effect of self-purification is considered here [9]. Self-purification is the ability of the river to regenerate itself as when the pollutant start oxidizing using DO, the atmospheric oxygen automatically starts entering the water through its surface and try to maintain the saturated level. It means that some amount of organic pollutant is eliminated by river itself. Also, due to the fact that pollutants are largely biological waste, pollutant concentration and Biological Oxygen Demand (BOD) concentration can be used interchangeably.

Let the time interval of the integration be $[t_0, t_{end}] = [0,20]$ and the number of steps in which the time interval is divided be $N = 200$. Then the time duration required per step is

$$\Delta t = \square = \left[\frac{t_{end} - t_0}{N}\right] = 0.1$$

At this point, the improved Runge-Kutta method of order four is applied on the system of equations (8) and (9). It has been implemented in MATLAB and the simulated graphics are generated.

VI. RESULT AND INTERPRETATION

The results are shown in Figure 1.

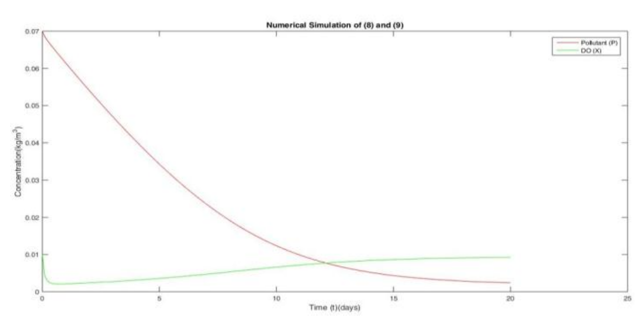

Figure 1

This can be interpreted as follows: At $t_0 = 0$, pollutant is discharged into the water with biochemical oxygen demand concentration 7 times the saturated oxygen concentration. This pollutant immediately starts reacting with the dissolved oxygen to produce harmless compounds. As a result both the DO and pollutant concentration start decreasing. After some time, the self-purification system of water becomes active and consequently pollutant concentration start reaching to zero and DO concentration again reaches its saturated value.

A. Discussion : The Kshipra River

The data for River Kshipra water quality status November 2018, made available by M.P. Pollution control board (MPPCB) on their official webpage is shown below (Table 2):

Table 2. Water Quality Index data (by MPPCB)

S.No.	Name of monitoring stations	DO (as per CPCB* calculator) (mg L^{-1})	BOD (as per CPCB* calculator) (mg L^{-1})
1.	Siddhwat	0.0068	0.014
2.	Ramghat	0.007	0.010
3.	Gaughat	0.0072	0.009
4.	1 km d/s Triveni	0.0072	0.007

*CPCB – Central Pollution Control Board, India.

Here, values are converted from mg L^{-1} to kg m^{-3}, using the relation

$$1\, mgL^{-1} = 0.001\, kg\, m^{-3}.$$

Due to continuous change in the width of a river, the parameter α also changes. These changes can be seen in the following Table 3:

Table 3. α for different monitoring stations of Kshipra River

S.No.	Name of monitoring stations	Width of the river*, w (m)	α (m^2 day^{-1})
1.	Siddhwat	85	4.675
2.	Ramghat	80	4.4
3.	Gaughat	65	3.575
4.	1 km d/s Triveni	60	3.3

* measured using Google Earth (approx.)

Taking the parametric values of α from Table 3, other parametric values from Table 1 and initial conditions from Table 2, the same model is employed ignoring the further addition of pollutant, i.e., $q = 0$, to get the time required by the DO to reach its saturation level at these monitoring locations. Following Figure 2 shows the numerical simulation results at different monitoring stations:

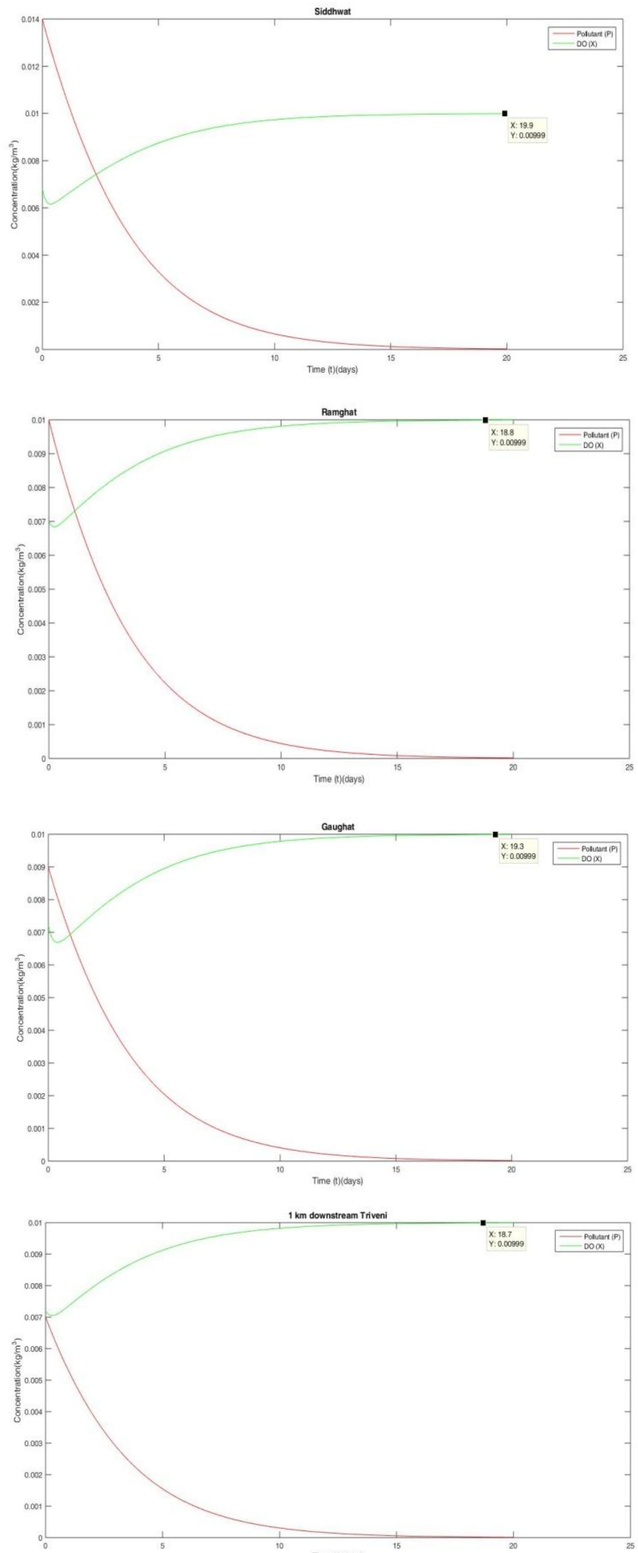

Figure 2

It can be seen that, at all monitoring stations almost 18-20 days is required for the pollutant to decay and DO to reach its saturation level. Furthermore, if the pollutant discharge is continuously going on, it is impossible for the river to self-purify itself under the absence the diffusion and flow.

VII. Conclusion

In this paper, a one-dimensional time-dependent mathematical model for the river pollution was presented by considering coupled partial differential equations and it was solved numerically. The results obtained signifies that increasing the DO concentration helps us to get clean water within a short time as it helps in rapid removal of biological pollutants. Furthermore, the results considering the case of River Kshipra were discussed. The standard parameters given by CPCB were used and data for the river was obtained by M.P. Pollution Control Board.

This study discovered that the river is in pathetic condition because of reduced flow and excessive pollutant discharge, which is far beyond its assimilative capacity. This condition can be treated or controlled by proper government planning, by increasing the flow, by promoting artificial re-aeration and by controlling the discharge of wastewater/pollutants. This model and its solution provide a tool for policy-development to handle the water-quality issues of the river. This model can be further solved for non-zero diffusion case or by taking variable area of the river.

References

[1]. B. Pimpunchat, W.L. Sweatman, G.C. Wake, W. Triampo, A. Parshotam, "*A Mathematical Model for Pollution in a River and its Remediation by Aeration*", Applied Mathematics Letters 22, Elsevier, pp.304-308, 2009.

[2]. H.W. Streeter, E.B. Phelps, "*A Study of the Pollution and Natural Purification of the Ohio river*", US Public Health Service, Public Health Bulletin, 146, pp.1-65, 1925.

[3]. S.C. Chapra, "*Surface Water Quality Modeling*", McGraw Hill, New York, pp.235-257, 1997.

[4]. B. Pimpunchat, W.L. Sweatman, W. Triampo, G.C. Wake, A. Parshotam, "*Modelling River Pollution and Removal by Aeration*", in: L. Oxley. D. Kulasiri (Eds.) MODSIM 2007 International Congress on Modelling and Simulation. Land, Water and Environmental Management: Integrated Systems for Sustainability. Modelling and Simulation Society of Australia and New Zealand, December 2007, pp.2431-2437, 2007.

[5]. S. Hasadsri, M. Maleewong, "*Finite Element Method for Dissolved Oxygen and Biochemical Oxygen Demand in an Open Channel*", Procedia Environmental Sciences, 13, pp.1019-1029, 2012.

[6]. T. Simon, P. R. Koya, "*Studying the Effects of Reaction, Diffusion and Reaeration on River Pollution Using Numerical Simulation of Diffusion-Reaction Equations*", IOSR Journal of Mathematics, Vol.13, Issue 3(11), pp.30-36, 2017.

[7]. M. Benedini, G. Tsakiris, "*Water Quatity Modelling for Rivers and Streams*", Springer Science + Business Media Dordrecht, pp.27-53, 2013.

[8]. S.A. Hussain, W.G. Atshan, Z.M. Najam, "*Mathematical Model for the Concentration of Pollution and Dissolved Oxygen in the Diwaniya River (Iraq)*". American Journal of Scientific Research, Issue 78, pp.33-37, 2012.

[9]. T. Simon, P. R. Koya, "*Modeling and Numerical Simulation of River Pollution using Diffusion-Reaction Equation*". American Journal of Applied Mathematics, Vol. 3, Issue 6, pp.335-340, 2015.

[10]. A.S. Wadi, M.F. Dimian, F.N. Ibrahim, "*Analytical solutions for one-dimensional advection-dispersion equation of the pollutant concentration*", Journal of Earth System Science. Vol.123, No.6, pp.1317-1324, 2014.

[11]. K.S. Kumar, H.L. Tiwari, R.V. Galkate, "*Water Availability Assessment in Shipra River*", International Journal of Research in Engineering and Technology, Vol.4, Issue 11, pp.126-130, 2015.

[12]. R.C. Gupta, A.K. Gupta, R.K. Shrivastava, "*Water quality management of river Kshipra (India)*", International Journal of Environmental Sciences, Vol.5, No.2, pp.438-446. 2014.

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