International Journal of Scientific Research in Mathematical and Statistical Sciences
Research Paper
Volume-5, Issue-6, pp.81-85, December (2018)
E-ISSN: 2348-4519

On Star Decomposition and Star Number of Some Graph Classes

J.K. Sebastian[1], J.V. Kureethara[2*], S.Naduvath[2], C. Dominic[2]

[1]Dept. of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, India
[2]Dept. of Mathematics, CHRIST (Deemed to be University), Bangalore, Karnataka, India

**Corresponding Author: frjoseph@christuniversity.in, Tel.: +00-919341094110*

Available online at: www.isroset.org

Received: 12/Nov/2018, Accepted: 05/Dec/2018, Online: 31/Dec2018

Abstract—Graph decomposition is a partition of graph into its subgraphs. Star decomposition is the decomposition of the graph into stars. In this paper we define a parameter, the star number of graphs, as the minimum number of end vertices of stars in a star decomposition of a graph. We determine this parameter for certain fundamental graph classes.

Keywords—Decomposition, star decomposition, pendant number, star number

I. Introduction

Let G be a graph. The *decomposition* of G is defined as the partition of the edge set of G into its subgraphs. Different types of decompositions of graphs are available in literature such as path decomposition, cycle decomposition, triangle decomposition and few papers are available in diamond decomposition [11]. Hamiltonian Decomposition is studied in [4]. A *star-decomposition* of a graph G is the partitioning of the given graph into stars (that is, $K_{1,n}$). The first attempt in decomposition of graphs into stars seems to be done by Ae, Yamamoto and Yoshida in an unpublished paper titled *Line-disjoint Decomposition of Complete Graph into Stars*[1]. They determined that a complete graph of order $3r; r > 1$ is 3- star decomposible. Following the steps, Cain determined the necessary and sufficient condition for a complete graph on rm or $rm + 1; r > 1$ vertices to be m-star decomposible when either r is even or m is odd [2]. Star decompositions of graphs in different aspects can be seen in [8], [17], [21] and various other articles. Labeling of star related graphs is studied by Sunoj and Varkey [16]. Continuous Monotonic Star Decomposition is studied in [18] and [19]. Diametral path decompositions are explored in [9].

The star decomposition problems are said to be NP-complete and have a wide range of applications such as scientific computing, parallel computing, distributed systems and is similar to the famous Master-slave paradigm where the master is the root vertex and the slaves are the pendant vertices [10]. The master assigns problems to slaves and collects the results. It is one of the natural questions that how to distribute the slaves economically. A new parameter for graphs is introduced recently in [12], which deals with the end vertices of graphs in a particular path decomposition of graphs namely, the pendant number of graphs. The *pendant number of graphs* denoted by $\Pi_p(G)$is the least number of end vertices of paths in a given path decomposition of a graph [12]. Further development of this parameter can be seen in [13], [14] and [15]. In this context, we define and introduce another parameter, namely star number of a graph and this parameter is discussed for some classes of graphs. We determine the bounds for this parameter. We, in this paper, determine the star number of paths, cycles, complete graphs, bipartite graphs and few classes of acyclic and cyclic graphs.

For terms and definitions in Graph Theory, we refer to Chartrand [3] and Harary [6]. Unless mentioned otherwise, all graphs we consider in this paper are undirected, finite, simple and connected. We have taken the convention of darkening the root vertices of stars to distinguish them from end vertices of stars in the star decomposition. If a vertex is already counted as an end vertex in a star decomposition, it remains as an end vertex even if it becomes the root vertex of any other stars in the same star decomposition. Section II is about major results in associated with star number. Section III is the conclusion and future scope of the work. This section also contains invitation for the readers to explore the open problems.

II. Star Number

Definition 1. The star number of a graph G, denoted by $\Pi_s(G)$, is the minimum number of vertices in the graph such that they are the pendant vertices of a star in a given star decomposition of the graph G. Let $V_s(G)$ be the set of all pendant vertices of stars in a star decomposition of G. Then, $\Pi_s(G) = \min\{|V_s(G)|: G \text{ has a star decomposition}\}$.

Let G be a graph. Then, the following proposition is obvious;

Proposition 2.$\Pi_s(K_{1,n}) = n$ (see Figure 1).

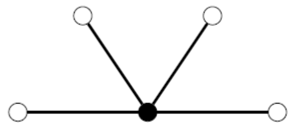


Figure 1. *A Star $K_{1,4}$.*

Proposition 3. Let $V(G)$ be the vertex set of a graph G.Then,$\Pi_s(G) = V(G)$ if and only if $G = K_2$ (see Figure 2).

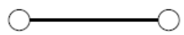


Figure 2. *K_2.*

Let $G = P_n$ or C_n.Then, alternate vertices starting from the first serves as a pendant vertex (see Figure 3 and Figure 4). Thus we have;

Proposition 4.

1. $\Pi_s(P_n) = \lceil\frac{n+1}{2}\rceil$.

$\Pi_s(C_n) = \lceil\frac{n}{2}\rceil$.

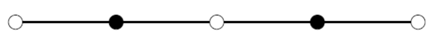


Figure 3. *A path graph P_5.*

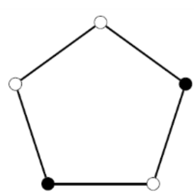


Figure 4. *A cycle graph C_5.*

Proposition 5. Let $G = K_n$ be a complete graph on n vertices.Then,$\Pi_s(K_n) = n - 1$.

Proof. Let $G = K_n$ be a complete graph on n vertices. Let v_1 be a vertex such that a star is rooted at v_1. Then, definitely, all other vertices become pendant vertices for the given star. Thus, leaving the lone vertex v_1 and taking all the remaining vertices as pendant we get, $\Pi_s(K_n) = n - 1$.

The bounds of the star number of graphs are described in the following result;

Proposition 6.

1. Let G be a graph on n vertices. Then,$2 \le \Pi_s(G) \le n - 1$.
2. Let p be the number of pendant vertices. Then, $p \le \Pi_s(G) \le n - 1$.

Proof.

1. Let G be a graph on n vertices. Since the smallest star is $K_{1,1}$ and its both ends are pendant vertices, star number of any graph must be always greater than or equal to 2. Again, in any star with n>2, its root vertex remains unaltered. Thus, at least one vertex can be set apart as the root vertex when $n > 2$. Hence, maximum value of star number is $n - 1$.

2. Let G be a graph on n vertices. Since pendant vertices of any graph G are *ipso facto* belong to the category of end vertices, they serve as lower bound. The upper bound is already proved in the previous case.

Proposition 7. For a complete bipartite graph $K_{m,n}$ we have, $\Pi_s(K_{m,n}) = m$, where $1 < m \le n$.

Proof. Let $K_{m,n}$ be a complete bipartite graph with $1 < m \le n$. Let $u_1, u_2, \dots, u_m$ be the vertices in one part and $v_1, v_2, \dots, v_n$ be the vertices in other part. Then, the least number of pendant vertices is obtained in a star decomposition when $v_1, v_2, \dots, v_n$ are taken as the root vertices. Hence, $\Pi_s(K_{m,n}) = m$.

The following theorem describes the pendant number of a binary tree.

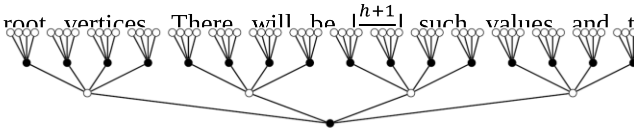
Theorem 8. Let T be a complete binary tree of height h. Then,$\Pi_s(T) = \Sigma_{i=0}^{\lfloor\frac{h+1}{2}\rfloor} 2^{h-2i}; i \le \frac{h}{2}$.

Proof. There are $2^{h+1} - 1$ pendant vertices on a complete binary tree T of height h. Since all the pendant vertices are counted in the calculation of star number, the vertices on height $h - 1$ can be treated as root vertices. Eventually, the verticeson height $h - 2$ too counted as pendant vertices of the star. Then, the vertices on height $h - 3$ becomes roots and so on. In short, from the pendant vertices on height $h$to the root of the binary tree, vertices in all the alternative heights counted for star number. There will be $\lfloor\frac{h+1}{2}\rfloor$ such heights and in each height $2^{h-2i}; i \le \frac{h}{2}$ vertices are there. Hence, $\Pi_s(T) = \Sigma_{i=0}^{\lfloor\frac{h+1}{2}\rfloor} 2^{h-2i}; i \le \frac{h}{2}$.

An$m - ary$tree is a tree all whose internal vertices have exactly m sons. A vertex is said to be at stage i, if its distance from the root vertex is i. An m-ary tree is said to be a complete m-ary tree if all its pendant vertices are at the same stage. In a similar way, we determine the star number of an m-ary tree as given in the following theorem.

Theorem 9. Let T be a complete m-ary tree of height h. Then, $\Pi_s(T) = \Sigma_{i=0}^{\lfloor\frac{h+1}{2}\rfloor} m^{h-2i}; i \le \frac{h}{2}$.

Proof. Let T be an m-ary tree with height h. Then, the number of vertices $n$is $\frac{m^{h+1}-1}{m-1}$. As in the case of binary tree, the vertices in the heights $h, h-2, \ldots$ are counted for the star number and the vertices of height $h-1, h-3, \ldots$ are for the root vertices. There will be $\lfloor\frac{h+1}{2}\rfloor$ such values, and the



$\frac{h}{2}$

$_{i=0}$ 2

Figure 5. *A 4-ary tree.*

A tree T is called a *caterpillar*, if removing of all its pendant vertices makes it a path (see [10]).

Theorem 10. For a caterpillar T on n vertices, $\lceil\frac{n+1}{2}\rceil \leq \Pi_s(T) \leq n-1$.

Proof. The number of caterpillars with $n \geq 3$ of unlabeled vertices can be counted as $2^{h-4} + 2^{\lfloor\frac{n-4}{2}\rfloor}$ (see [14]). Hence, a caterpillar on n vertices can be varied from a path P_n to a star $K_{1,n-1}$. These are the extreme cases of a caterpillar. It is already found the star number of path P_n and the star $K_{1,n-1}$ as $\lceil\frac{n+1}{2}\rceil$ (see Proposition 4) and $n-1$ (see Proposition 2 respectively (see Figure 6). Hence, the result.

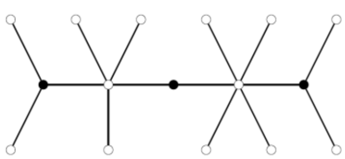


Figure 6. *A caterpillar.*

The *friendship graph* is obtained by joining n copies of the cycle c_3 to a common vertex (see [5]).

Proposition 11. For friendship graphs F_n with $2n+1$ vertices, $\Pi_s(F_n) = n+1$.

Proof. Let F_n be a friendship graph with $2n+1$ vertices. There are n triangles whose one vertex is common to all. Since the star number of a triangle (that is, cycle C_3) is $\lceil\frac{3}{2}\rceil = 2$, each triangle has a root vertex. Take the root vertex as one of the vertices other than the common one.There are $n+1$such vertices counted for the star number (see Figure 7).

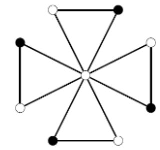


Figure 7. *A frienship graph F_4.*

A *pineapple graph*, denoted by K_n^m, is a graph obtained by appending m pendant edges to a vertex of a complete graph K_n; $m \geq 1, n \geq 3$ (see [20]).

Proposition 12. For a pineapple graph K_n^m, $\Pi_s(K_n^m,) = n+m-1$.

Proof. In a pineapple graph K_n^m, only the root vertex is merged to become a single root and all other vertices remain as pendant, its star number is the total number of vertices minus one, that is, $n+m-1$ (see Figure 8).

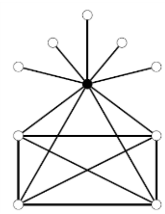


Figure 8. *A pineapple graph K_5^5.*

By the *one-point union* of a collection of graphs (possibly with different order), we mean a graph obtained by replacing some or all edges of a path P by some graphs in the collection. A *pan graph* is the one point union of a cycle C_n and a K_2.

Proposition 13. For a pan graph G, $\Pi_s(G) = \lceil\frac{n+1}{2}\rceil$.

Proof. Let G be a pan graph such that u_1, u_2 be the vertices of K_2 and $v_1, v_2, \ldots, v_n$ be the vertices of $C_n$with $u_1 = v_1$. Then, the star number of the pan graph is one more than the star number of the cycle C_n. That is, $\lceil\frac{n}{2}\rceil + 1$ and hence, $\Pi_s(G) = \lceil\frac{n+1}{2}\rceil$ (see Figure 9).

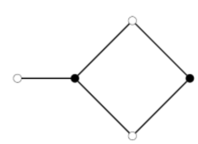


Figure 9. *A Pan graph.*

A *wheel graph* W_n is the join $K_1 + C_n$, where the vertex K_1 is called the hub of the wheel graph. The next result is on the star number of a wheel graph.

Proposition 14. For a wheel graph W_n, $\Pi_s(W_n) = \lceil\frac{n+2}{2}\rceil$.

Proof. In a wheel graph its hub and all the alternate vertices are part of the star number. Thus, $\Pi_s(W_n) = \lceil\frac{n+2}{2}\rceil$ (see Figure 10).

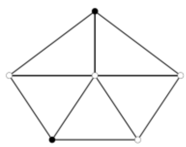


Figure 10. *A wheel graph W_5.*

Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the set of internal vertices of a caterpillar T with the vertices $v_2, v_3, \ldots v_{n-1}$ having degree 3 and the remaining two vertices v_1 and v_n having degree 2. Then, T is called a *comb graph*(see [15]).

Theorem 15. For a comb graph G on $2n$ vertices, $\Pi_s(G) = n + \lfloor\frac{n}{2}\rfloor$.

Proof. In a comb graph G on $2n$ vertices, n pendant vertices are definitely end vertices of some stars in the star decomposition of G (see Proposition 6). In addition, the alternate vertices among the internal vertices of G are also end vertices of some stars and the number of such vertices is same as $\lfloor\frac{n}{2}\rfloor$. Thus, the star number of comb graph on $2n$ vertices be $n + \lfloor\frac{n}{2}\rfloor$ (see Figure 11).

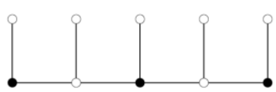


Figure 11. *A comb graph.*

The *banana tree* $B_{n,k}$ is obtained as follows: Let $K_{1,k-1}$ be a k- star and v be an isolated vertex. Take n copies of k- star and connect one pendant vertex each from these stars and join to v. It has $nk + 1$ vertices and nk edges.

Theorem 16. For a banana tree $B_{n,k}$, the pendant number is $n(k-1)$.

Proof. Let $B_{n,k}$ be a banana tree on n copies of a k- star. Then, the minimum cardinality of end vertices in a star decomposition is obtained by taking the roots of each star and the vertex v as the root vertices in the required star decomposition. Hence, $\Pi_s(B_{n,k}) = n(k-1)$ (see Figure 12).

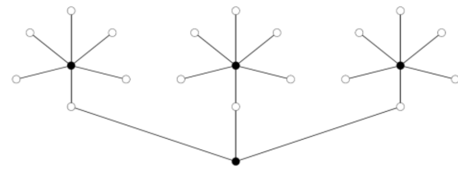


Figure 12. *A banana tree $B_{3,7}$.*

III. CONCLUSION AND FUTURE SCOPE

In this study, we introduce a new concept known as the star number of graphs and determined this parameter for few graph classes, especially for trees. We propose its bounds for arbitrary graphs. Finding the star number of many other classes of graphs remains open. Comparison of star number with other graph parameters such as pendant number, domination number, diameter, path decomposition of graphs of length two, etc. are promising. Since star decomposition is NP-complete, finding the star number may help to get some more clarity about the nature of such graphs.

ACKNOWLEDGMENT

The first author wishes to thank the Research Centre, Department of Mathematics, CHRIST (Deemed to be University) for all the facilitations in this research endeavour.

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AUTHORS PROFILE

Mr. J.K. Sebastian is a research Scholar in Manonmaniam Sundaranar University, Tirunelveli, Tamilnadu, India. He is a member of ADMA and MTA. He pursed M.Sc (Mathematics) from St. Joseph's College (Bharathidasan University), Trichy, Tamilnadu and M.Ed from PSG College, Kankayam, Tamilnadu. He has has interests in the areas of Discrete and Combinatorial Mathematics.

Dr. J. V. Kureethara received his PhD in Mathematics from Manonmaniam Sundarnar University, Tirunelveli, India in 2010. He has MSc Mathematics and MA Economics from Madras University. He is currently an Associate Professor in the Department of Mathematics, CHRIST (Deemed to be University), Bangalore, India. He is the Mathematics issue editor of *Mapana* Journal of Sciences and reviewer of many Mathematics journals. He is the author over fifgy articles in the fields of Graph Theory, Number Theory, Church History, Sacred Liturgy and Sports both in English and Malayalam. He is an active blogger and his blog has a total of more than 120 thousand pageviews. He co-edited two books.

Dr. N. K. Sudev is an Associate Professor of Mathematics in CHRIST (Deemed to be University), Bangalore, India. He has eighteen years of teaching experience and seven years of research experience. His primary research areas are graph theory and combinatorics. He has authored three books and more than eighty research publications. He is an active member of eleven professional societies and member of editorial boards of eleven journals. He is also a referee of four prestigious reviewing services and a reviewer of more than forty international research journals.

Dr. C. Dominic pursed Ph D in Mathematical Sciences from University of Mysore in 2011. He is currently working as Assistant Professor of Mathematics in CHRIST (Deemed to be University), Bangalore, India. He is a member of ADMA since 2017. He has published 12 research papers in reputed international journals including Discrete Mathematics, Disscussiones Mathematica Graph Theory. His research area includes graph colorings, cop-robber game in graphs and the zeroforcing number of a graph. He has 7 years of teaching experience and 2 years of post doctoral experience from University of Aveiro, Portugal and 5 years of research experience.

