International Journal of Scientific Research in Mathematical and Statistical Sciences
Vol.**11**, Issue.**1**, pp.**01-05**, February **2024**
E-ISSN: 2348-4519
Available online at: www.isroset.org

Research Article

Ellipse Perimeter Approximation: Two High Accuracy Formulae Derived From a Perimeter Property

K. Idicula Koshy[1*]

Corresponding Author: kiduculakoshy@gmail.com

Received: 22/Dec/2023; **Accepted**: 20/Jan/2024; **Published**: 29/Feb/2024. | **DOI**: https://doi.org/10.26438/ijsrmss/v11i1.15

Abstract— There is no simple formula for the **exact perimeter** of an ellipse; but there are several formulae giving approximate value of the perimeter. However, some of them fail to give the correct perimeter for degenerate ellipse; some others, though simple, provide very low accuracy level. In this research article the author introduces two high accuracy formulae derived from a geometric property of the elliptic arc in the first quadrant. Both these formulae are applicable to ellipses of all **aspect ratios** and the absolute relative error yields are less than one centimeter per kilometer.

Keywords—Ellipse, Aspect Ratio, Perimeter, Simpson's Rule, p-norm, Absolute Relative Error, ppm.

1. Introduction

The ellipse, whose rectangular cartesian equation is $(x/a)^2 + (y/b)^2 = 1$, is named here as the **standard ellipse**. '$a \neq 0$' and 'b' ($a \geq b \geq 0$) are the major and minor radii of the ellipse. It is well-known from Integral Calculus that the perimeter P (a, b) is given by

$$P(a, b) = \int_0^{2\pi} \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}\, d\theta$$

where ($a \cos \theta$, $b \sin \theta$), $0 \leq \theta < 2\pi$, is a parametric point on the ellipse. Due to the symmetry of the ellipse w. r. t. its axes, **P (a, b) = 4* Q (a, b),** where Q (a, b) is the arc-length of the standard ellipse in the first quadrant (also called **Quarter Perimeter** in this article). Therefore, the first-quadrant arc-length (Quarter perimeter) is given by the definite integral:

$$Q(a, b) = \int_0^{\pi/2} \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}\, d\theta$$

Obviously, **Q (a, a) = πa /2; P (a, a) = 2π*a;**
Q (a, 0) = a; and P (a, 0) = 4*a.
(If b = 0, the elliptic arcs above and below the x-axis degenerate (shrink) to line segments of length 2*a each). If 'a' ≠ 'b' ≠ 0, the integral for P (a, b) or Q (a, b) given above could not be evaluated till now by any known direct integration methods. Therefore, a closed formula is not available for Ellipse Perimeter, which makes it necessary to look for Numerical Integration Methods to approximate Q (a, b). We use here values obtained by Simpson's 1/3-Rule method for the Numerical Integration of the integral for Q (a, b), yielding Absolute Relative Error less than $10^{(-10)}$ in place of Q (a, b) values, for computation of Absolute Relative Error due to our formulae. These Simpson values, obtained with a = 100, and 'b' ranging from 0 to 100, are denoted here by **Q (100, b; Sim),** and are obtained by dividing the interval of integration [0, π /2] into 500 equal sub-intervals, so that the **length h** of each sub-interval is equal to π/1000. Then the Absolute Relative Error, which is of order h^4, is less than $10^{(-10)}$ [1]. Therefore, the error due to replacing Q (100, b) by Q (100, b: Sim) is very negligible.

Section 2 introduces the Terminology, Notations and Abbreviations.
Section 3 gives the two new formulae for Ellipse Perimeter Approximation. Each formula is immediately followed by the relevant table that gives the Relative Error due to the formula. In the tables a few rows are deleted without affecting the Result n any way, in order to limit the number of pages.
Section 4 gives information on the Materials and Methods used.
In Section 5, the Discussion and Comments are presented.

2. Terminology, Notations and Abbreviations

Conventional terminologies and notations related to the standard ellipse are used in this article. The lengths of the semi-major axis (**major radius**) and the semi-minor axis (**minor radius**) of the standard ellipse are denoted by '$a \neq 0$' and 'b' respectively. The ratio (b/a) is called the **Aspect Ratio,** which takes values in the closed interval [0, 1]. **P (a, b)** and **Q (a, b)** denote respectively the total perimeter and the Quarter Perimeter of the standard ellipse.

S = Q (a, b; Sim) denotes the Quarter Perimeter values obtained by Simpson's (1/3) Rule as mentioned above. **AM** = (a + b)/2 and **GM** = $\sqrt{a * b}$ are respectively the Arithmetic and the Geometric means of 'a' and 'b'**. EPM**, **QPM, RE** and **ARE** respectively the abbreviations used for Ellipse Perimeter, Quarter Perimeter, Relative Error and Absolute Relative Error. **'ppm'** stands for 'particles per million', which is same as millimeter per kilometer.

3. Results

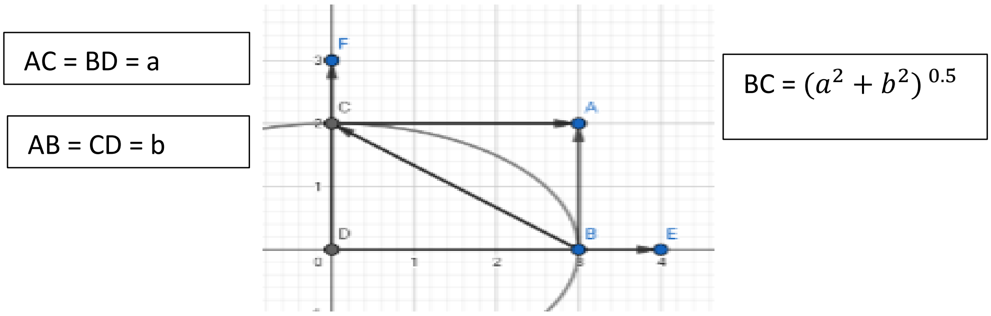

Figure 1. First Quadrant Ellipse

It is obvious from Figure 1 that $(\boldsymbol{a^2 + b^2})^{\left(\frac{1}{2}\right)} \leq \boldsymbol{Q}\ (\boldsymbol{a}, \boldsymbol{b}) \leq (\boldsymbol{a} + \boldsymbol{b})$. **This is the property of the (quarter) perimeter we are using derive the formulae.**

Formula 1: 'p-norm formula'

$$Q\ (a, b) \approx Q\ (a, b; p) = (a^p + b^p)^{(1/p)} \quad \ldots\ldots\ldots \quad (1)$$

where $p = p\ (a, b; k) = \ln(2)/\ln(\pi/2) + 1 - (b/a)^{k}$; .... (1a)

$$k = 0.03214 - 0.0734*(b/a) + 0.0863*(b/a)^2 - 0.0681*(b/a)^3 + 0.02306*(b/a)^4 \quad \ldots. \quad (1b)$$

Table 1: Relative Error due to Formula 1: 'p-norm formula'

a	b	b/a	S= Q (a, b; Sim)	(Estimate.) k = 0.03214 -0.0734*x+0.0863*x^2-0.0681*x^3+0.02306*x^4), x=(b/a)	p = ln (2)/ln (π/2) + (1-(b/a) ^{k})	Q (a, b; p)	Rel. Error = (Q (a, b; p) -S)/S
100	100	1.00	157.079632679490	0.0000000000000	1.5349285356614	157.079632679490	1.80938222E-16
100	99	0.99	156.295221198759	0.0001306120906	1.5349298483559	156.295229764799	5.48067896E-08
100	98	0.98	155.512803035381	0.0002651545696	1.5349338924872	155.512835057440	2.05912686E-07
100	97	0.97	154.732408602913	0.0004034908986	1.5349408255989	154.732475827253	4.34455461E-07
100	96	0.96	153.954068977126	0.0005454900736	1.5349508034062	153.954180290926	7.23032530E-07
100	95	0.95	153.177815915124	0.0006910266250	1.5349639800653	153.177977621101	1.05567491E-06
100	94	0.94	152.403681875157	0.0008399806176	1.5349805084506	152.403897956056	1.41781941E-06
100	93	0.93	151.631700037168	0.0009922376506	1.5350005404427	151.631972410010	1.79627902E-06
100	92	0.92	150.861904324107	0.0011476888576	1.5350242272262	150.862233084084	2.17921137E-06
100	91	0.91	150.094329424045	0.0013062309066	1.5350517195980	150.094713077964	2.55608537E-06
100	90	0.90	149.329010813121	0.0014677660000	1.5350831682871	149.329446502294	2.91764588E-06
100	89	0.89	148.565984779372	0.0016322018746	1.5351187242866	148.566468491854	3.25587639E-06
100	88	0.88	147.805288447480	0.0017994518016	1.5351585391972	147.805815219576	3.56395973E-06
100	87	0.87	147.046959804474	0.0019694345866	1.5352027655855	147.047523911435	3.83623682E-06
100	86	0.86	146.291037726448	0.0021420745696	1.5352515573552	146.291632862276	4.06816328E-06
100	85	0.85	145.537562006337	0.0023173016250	1.5353050701339	145.538181452637	4.25626410E-06
Rows Deleted							
100	40	0.40	115.065562978324	0.0128199360000	1.5466066000402	115.066425007502	7.49163481E-06
100	39	0.39	114.493147277843	0.0131340855946	1.5472195436896	114.494050965188	7.89293828E-06
100	38	0.38	113.927062714469	0.0134537691616	1.5478618246285	113.928000134019	8.22824294E-06
100	37	0.37	113.367475203505	0.0137791822266	1.5485351012379	113.368437323081	8.48673374E-06
100	36	0.36	112.814557442816	0.0141105258496	1.5492411584812	112.815534199862	8.65807630E-06

100	35	0.35	112.268489320333	0.0144480066250	1.5499819204264	112.269469721703	**8.73264953E-06**
100	34	0.34	111.729458356001	0.0147918366816	1.5507594644195	111.730430602986	8.70179628E-06
100	33	0.33	111.197660182088	0.0151422336826	1.5515760371903	111.198611821779	8.55809097E-06
100	32	0.32	110.673299066356	0.0154994208256	1.5524340732315	110.674217170214	8.29562203E-06
100	31	0.31	110.156588483269	0.0158636268426	1.5533362158636	110.157459853462	7.91028667E-06
100	30	0.30	109.647751739223	0.0162350860000	1.5542853414912	109.648563142918	7.40009423E-06
100	29	0.29	109.147022658760	0.0166140380986	1.5552845876721	109.147761090064	6.76547363E-06
100	28	0.28	108.654646339879	0.0170007284736	1.5563373857701	108.655299308529	6.00957872E-06
100	27	0.27	108.170879987965	0.0173954079946	1.5574474991528	108.171435833134	5.13858414E-06
100	26	0.26	107.695993839586	0.0177983330656	1.5586190681414	107.696442066246	4.16196224E-06
100	25	0.25	107.230272189460	0.0182097656250	1.5598566632460	107.230603823663	3.09272928E-06
Rows Deleted							
100	19	0.19	104.646409451062	0.0208723841226	1.5689980609161	104.645984873418	-4.05725955E-06
100	18	0.18	104.254261485833	0.0213511682656	1.5708793370595	104.253730706326	-5.09120203E-06
100	17	0.17	103.874232134835	0.0218407546426	1.5728900994394	103.873610813282	-5.98147915E-06
100	16	0.16	103.506806897050	0.0223414550016	1.5750442488044	103.506114699686	-6.68745742E-06
100	15	0.15	103.152507352688	0.0228535866250	1.5773580974415	103.151767798253	-7.16952456E-06
100	14	0.14	102.811895824480	0.0233774723296	1.5798509964413	102.811135917604	**-7.39123493E-06**
100	13	0.13	102.485580990886	0.0239134404666	1.5825461881863	102.484830584827	-7.32206474E-06
100	12	0.12	102.174224732294	0.0244618249216	1.5854719901300	102.173515552104	-6.94089133E-06
100	11	0.11	101.878550604060	0.0250229651146	1.5886634796827	101.877914849602	-6.24031706E-06
100	10	0.10	101.599354502522	0.0255972060000	1.5921649619569	101.598822939709	-5.23195069E-06
100	9	0.09	101.337518361821	0.0261848980666	1.5960337076668	101.337117803295	-3.95271695E-06
100	8	0.08	101.094028165077	0.0267863973376	1.6003458464591	101.093778245671	-2.47214806E-06
100	7	0.07	100.869998319400	0.0274020653706	1.6052061220433	100.869907503922	-9.00321996E-07
100	6	0.06	100.666705836685	0.0280322692576	1.6107650465688	100.666766697293	6.04575339E-07
100	5	0.05	100.485640478649	0.0286773816250	1.6172514948497	100.485824543383	1.83175161E-06
100	4	0.04	100.328582826687	0.0293377806336	1.6250413626290	100.328835987330	2.52331525E-06
100	3	0.03	100.197736240671	0.0300138499786	1.6348248380001	100.197977580767	2.40863821E-06
100	2	0.02	100.095979045020	0.0307059788896	1.6481167363367	100.096111621356	1.32449212E-06
100	1	0.01	100.027463597804	0.0314145621306	1.6696202231692	100.027422180597	-4.14058351E-07
100	0	0.00	100.000000000000	0.032140000000000	2.5349285356614	100.000000000000	4.26325641E-16

Note: The Maximum ARE is less than $8.73265*10^{(-6)}$, which is less than 9 ppm.

A formula similar in appearance to Formula 1 is available in the literature for QPM approximation. It is the Y-NOT formula, by Roger Maertens, given in Numericana [2] . It stats that:

$\mathbf{Q\,(a,b) \approx (a^{Y} + b^{Y})^{(1/Y)}}$,

where Y = ln (2)/ln (π/2). Though the formula is exact for (b/a) = 1 and (b/a) = 0, the maximum ARE is of order $10^{(-3)}$, which is much higher than that yielded by Formula 1. The index ‘p’ used in Formula 1 is a fourth-degree polynomial in (b/a), where as it is a pure constant in Y-NOT formula. The much lower ARE by Formula 1 justifies the choice of a polynomial for ‘p’ rather than a simpler constant.

The qualifier **‘p-norm’** is used in Formula 1 above, as $(a^p + b^p)^{(1/p)}$ is called the p-norm of the ordered pair (a, b) in Mathematical Analysis.

Now we derive the second formula noting that $(\boldsymbol{a^2} + \boldsymbol{b^2})^{(\frac{1}{2})} \le \boldsymbol{Q\,(a,b)}$. The author has established that the definite integral form for Q (a, b), given in Section 1 (Introduction), satisfies a first order linear partial differential equation and its solution is a linear combination of $(a^p + b^p)^{(1/p)}$ and $\sqrt{a*b}$ [3]. One may try suitable functions for p, taking values not less than ln(2)/ln(π/2), keeping in mind that Q (a, b) is a 1-dimensional symmetric function of ‘a’ and ‘b’, and, that its value is ‘π/2*a’ if ‘b’ = ‘a’ and ‘a’ if ‘b’ = 0.

Now, to derive Formula 2 using the perimeter property, we fix p by taking p = 2 and consider a linear combination of $(a^2 + b^2)^{(1/2)}$ and $\sqrt{a*b}$, that fulfils the basic requirements and minimizes the ARE to the extent possible, to approximate Q (a, b).

Formula 2: ‘2-norm and GM based formula’.

$$Q\,(a,b) \approx Q\,(a, b; 2, k) := (a^2 + b^2)^{(1/2)} + (\pi/2 - 2^{(1/2)}) * ((GM/AM)^{k}) * (a*b)^{(1/2)} \quad \ldots.. \quad (2)$$

$$k = 2.6071 + 1.2243*(b/a) - 1.2673*(b/a)^2 + 0.45566*(b/a)^3. \quad \ldots\ldots \quad (2a)$$

Table 2: Relative Error due to Formula 2

a	b	b/a	S = Q (a, b; Sim)	GM/AM	k = 2.6071 + 1.2243*(b/a) - 1.2673*$(b/a)^2$ + 0.45566*$(b/a)^3$)	Q (a, b; 2, k) = $(a^2+b^2)^{0.5}$ + (pi/2- $2^{0.5}$) * ($(GM/AM)^k$) * $(a*b)^{0.5}$	Relative Error (Formula2- S) / S
100	100	1.00	157.0796326795	1.0000000000	3.0197600000	157.0796326795	0.000000E+00
100	99	0.99	156.2952211988	0.9999873740	3.0192027123	156.2952205397	-4.216810E-09
100	98	0.98	155.5128030354	0.9999489835	3.0186626267	155.5128007799	-1.450344E-08
100	97	0.97	154.7324086029	0.9998840408	3.0181370092	154.7324043159	-2.770598E-08
100	96	0.96	153.9540689771	0.9997917317	3.0176231258	153.9540626374	-4.117911E-08
100	95	0.95	153.1778159151	0.9996712149	3.0171182425	153.1778078323	-5.276786E-08
100	94	0.94	152.4036818752	0.9995216201	3.0166196254	152.4036726108	-6.078806E-08
100	93	0.93	151.6317000372	0.9993420478	3.0161245406	151.6316903319	-6.400561E-08
100	92	0.92	150.8619043241	0.9991315674	3.0156302541	150.8618950288	-6.161442E-08
100	91	0.91	150.0943294240	0.9988892161	3.0151340319	150.0943214371	-5.321315E-08
100	90	0.90	149.3290108131	0.9986139979	3.0146331400	149.3290050220	-3.878077E-08
100	89	0.89	148.5659847794	0.9983048817	3.0141248445	148.5659820085	-1.865086E-08
100	88	0.88	147.8052884475	0.9979608000	3.0136064115	147.8052894105	6.515239E-09
100	87	0.87	147.0469598045	0.9975806474	3.0130751070	147.0469650623	3.575635E-08
100	86	0.86	146.2910377264	0.9971632791	3.0125281970	146.2910476509	6.784053E-08
100	85	0.85	145.5375620063	0.9967075089	3.0119629475	145.5375767486	1.012951E-07
100	84	0.84	144.7865733828	0.9962121076	3.0113766246	144.7865928476	1.344379E-07
100	83	0.83	144.0381135702	0.9956758010	3.0107664944	144.0381373954	1.654086E-07
100	82	0.82	143.2922252901	0.9950972679	3.0101298229	143.2922528312	1.922024E-07
100	81	0.81	142.5489523035	0.9944751381	3.0094638761	142.5489826240	2.127026E-07
Rows Deleted							
100	70	0.70	134.5592245368	0.9843059136	2.9994243800	134.5591451159	-5.902297E-07
100	69	0.69	133.8511777909	0.9830324098	2.9981938809	133.8510702223	-8.036438E-07
100	68	0.68	133.1464274484	0.9816918156	2.9968985651	133.1462882741	-1.045272E-06
100	67	0.67	132.4450361480	0.9802817691	2.9955356986	132.4448619565	-1.315199E-06
100	66	0.66	131.7470682676	0.9787998078	2.9941025474	131.7468557423	-1.613131E-06
Rows Deleted							
100	45	0.45	118.0172311018	0.9252695079	2.9429287675	118.0161320312	-9.312797E-06
100	44	0.44	117.4154359143	0.9212846640	2.9392576614	117.4143411404	**-9.323935E-06**
100	43	0.43	116.8192317748	0.9171242691	2.9354533896	116.8181504588	-9.256318E-06
100	42	0.42	116.2287558112	0.9127803801	2.9315132181	116.2276975495	-9.104991E-06
100	41	0.41	115.6441503067	0.9082445727	2.9274344129	115.6431250394	-8.865708E-06
Rows Deleted							
100	20	0.20	105.0502226984	0.7453559925	2.8049132800	105.0511818028	9.129960E-06
100	19	0.19	104.6464094511	0.7325880577	2.7970928419	104.6473853603	**9.325778E-06**
100	18	0.18	104.2542614858	0.7190916419	2.7890708891	104.2552285792	9.276296E-06
100	17	0.17	103.8742321348	0.7048043804	2.7808446876	103.8751629022	8.960522E-06
100	5	0.05	100.4856404786	0.4259177100	2.6652037075	100.4849252349	-7.117871E-06
100	4	0.04	100.3285828267	0.3846153846	2.6540734822	100.3279409096	-6.398147E-06
100	3	0.03	100.1977362407	0.3363205452	2.6427007328	100.1972742886	-4.610404E-06
100	2	0.02	100.0959790450	0.2772967769	2.6310827253	100.0957860442	-1.928158E-06
100	1	0.01	100.0274635978	0.1980198020	2.6192167257	100.0275252181	6.160342E-07
100	0	0.00	100.0000000000	0.0000000000	2.6071000000	100.0000000000	0.000000E+00

Maximum Absolute Relative Error is less than 9.32578*$10^{(-6)}$, which is less than 1cm per km

4. Materials and Methods

All Tables are prepared using MS Excel. Q (a, b; Sim) values used here are derived with step-width h = $\pi/1000$. Due to their very high accuracy, Q (a, b, Sim) values are used for computing not only the values of the index 'p' in the p-norm formula, but also the Relative Error due to the new formula. The process of computation of the index 'p' is given below.

i) Equate S = Q (a, b; Sim) with Q (a, b; p). Then $(a^p + b^p) = S^p$ or $1 + (b/a)^p = (S/a)^p$ **.** Solving this equation, we get 'p'. The point of intersection of the functions f (x) = $1 + (b/a)^x$ and g (x) = $(S/a)^x$ gives the value of 'p' for each (b/a). These p-values are used to get the best-estimate polynomial form of p (a, b; k) used in Formula 1, with very minor changes in some coefficients.

5. Discussion/Comments

The author has critically examined several Ellipse Perimeter Approximation Formulae known after several eminent Mathematicians, namely, Kepler, Euler, Seki, Muir, Maertens (Y-NOT formula), Rivera, Lindner, Zafary, Cantrell etc. and, in particular, the second formula of the Great Indian Mathematical Genius Srinivasa Ramanujan[2]. None of these formulae gives Maximum ARE less than 9-millimeter per kilometer consistently for ellipses of all aspect ratios. Therefore, both the formulae introduced here are better in accuracy for EPM approximation. Besides, EPM computations are possible with a scientific calculator.

The author has published a p-norm formula with different p [4]. In that formula 'p' contains an index which is a linear function of **ln(b/a).** Hence, it is not applicable for all (b/a) and 'p' needs to be redefined for small (b/a) from some point on. Such a predicament does not arise in the definition of 'k' in Formula 2. Besides, the coefficients in 'k' have some rhythm for easy recollection.

6. Conclusion and Future Scope

The Article provides two formulae with Absolute Relative Error to the Order of $10^{(-6)}$. There is much scope for formulae with higher accuracy and even for an exact formula for Ellipse Perimeter.

Data Availability
There is no data to provide other than that given in the Article.

Conflict of Interest.
There is no Conflict of Interest due to this Article.

Funding Sources
The Research leading to this Article is funded by Author's own resources

Author's Contribution
The entire Article, including the formulae in it, is contributed by the author and nobody else has any role in its preparation. It is prepared from my own knowledge of the English Language and as such there is no room for any charge of plagiarism.

Acknowledgements
The author acknowledges the encouragement got from certain Math Publications asking for Research Articles for publication in their Journals.

References

[1] Erwin Kreiszig, 'Advanced Engineering Mathematics' Ch. 17; Wiley, **2010.**
[2] Gerard P Michon, 'Perimeter of an ellipse: A Review', Weblink: Numericana
[3] K.I. Koshy, 'A new Approx. Formula by a new Approach' IJESIRD, July, Vol.**6**, Issue.**1**, pp.**1-6, 2019.**
[4] K. Idicula Koshy, 'Ellipse Perim. Approx. by p-norm formula', BOMSR, Vol.**11**, Issue.**4**, pp.**93-97, 2023.**

AUTHOR'S PROFILE

The author of this Research Article, **Dr. K. Idicula Koshy**, holds an M. Sc Degree in Mathematics from University of Kerala and Doctorate Degree (Dr. rer. nat.) from University of Dortmund (Now Technische Universtaet), Germany. His field of specialisation is 'Banach Lattices and Positive operators'. He has more than 40 years of experience of teaching Pure & Applied Mathematics and Statistics in Arts & Science Colleges, Engineering colleges and Universities in India and abroad. He was UGC Professor of Mathematics in the Agricultural Engineering College of Kerala Agricultural University. After retirement he has served in two very reputed engineering colleges in Kerala as Professor/ Vice Principal/ Principal. He is a Life Member of Ramanujan Mathematical Society, Chennai and also of Kerala Maths Association.

