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Research Paper

Putting Forward Another Generalization Of The Class Of Exponential Integrals And Their Applications

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Abstract—The central aim of this paper is to introduce another generalization to the well-known integral from the class of exponential integrals. For a better understanding of the main result presented here. the knowledge of elementary integrations of continuous functions, definite and indefinite integration, the Concept of Mathematical Induction, and Basic Algebraic Manipulation are pre-requesting. Further on the basis of providing generalization, authors also give some applications that can show the generalization's versatility, such as alternative proofs of the properties of Laplace Transform, Integration of e^ax sinbx and e^ax cosbx etc.

Keywords— Elementary Integration, Definite Integral, Indefinite Integration, Exponential class of Integration, Upper and Lower Bounds, Laplace Transform, Integration by Parts etc.

1. Introduction

In the papers [1][2] Toyesh have dealt with the generalization of $\int e^x(f(x) + f'(x)dx = e^x f(x) + c$ this generalization is one of popular integral form from the class of exponential integral and almost every student of mathematics has understood it in his/her intermediate mathematics book like [3]. The main result provided in the paper [1] is given below with an alternative and short proof

Theorem for $n \in \mathbb{N}$.

$$\int e^x\left(f(x) - (-1)^n \frac{d^n f(x)}{dx^n}\right)dx = e^x \sum_{k=1}^{n} (-1)^{k-1} \frac{d^{k-1} f(x)}{dx^{k-1}} + C$$

Proof

As $\int e^x\left(f(x) + f'(x)\right)dx = e^x f(x) + c_1$ And also $\int e^x\left(f'(x) + f''(x)\right)dx = e^x f'(x) + c_2$ Subtracting of both gives

$$\int e^x\left(f(x) + f'(x)\right)dx - \int e^x\left(f'(x) + f''(x)\right)dx = e^x f(x) - e^x f'(x) + c_1 - c_2$$

$$\Rightarrow \int e^x\left(f(x) - f''(x)\right)dx = e^x\left(f(x) - f'(x)\right) + c$$

Using $\int e^x\left(f(x) + f'(x)\right)dx = e^x f(x) + c_1$ we can say that for $f''(x)$

$$\int e^x\left(f''(x) + f'''(x)\right)dx = e^x f''(x) + c_3$$

Now, addition of $\int e^x\left(f(x) - f''(x)\right)dx$ and $\int e^x\left(f''(x) + f'''(x)\right)dx$ gives

$$\int e^x\left(f(x) - f''(x)\right)dx + \int e^x\left(f''(x) + f'''(x)\right)dx = e^x\left(f(x) - f'(x)\right) + e^x f''(x) + c_3 + c$$

$$\Rightarrow \int e^x\left(f(x) + f'''(x)\right)dx = e^x\left(f(x) - f'(x) + f''(x)\right) + C$$

Similarly, $\int e^x\left(f'''(x) + f''''(x)\right)dx = e^x f'''(x) + c_4$ and subtracting it with the above equation gives

$$\int e^x\left(f(x)+f''''(x)\right)dx-\int e^x\left(f'''(x)+f''''(x)\right)dx=e^x\left(f(x)-f'(x)+f''(x)\right)-e^x f'''(x)+C-c_4$$

$$\Rightarrow\int e^x\left(f(x)-f''''(x)\right)dx=e^x\left(f(x)-f'(x)+f''(x)-f'''(x)\right)+c$$

Using the above method for nth order derivative of $f(x)$ we can say that

$$\int e^x\left(f(x)-(-1)^n\frac{d^n f(x)}{dx^n}\right)dx=e^x\left(f(x)-f'(x)+f''(x)-f'''(x)+\cdots+(-1)^{n-1}\frac{d^{n-1}f(x)}{dx^{n-1}}\right)+C$$

Or

$$\int e^x\left(f(x)-(-1)^n\frac{d^n f(x)}{dx^n}\right)dx=e^x\sum_{k=1}^{n}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}+C$$

With following similar proof one can also prove

$$\int e^{-x}\left(f(x)-\frac{d^n f(x)}{dx^n}\right)dx=-e^{-x}\sum_{k=1}^{n}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

And this theorem was published in [2].

2. Overview

This paper contains the following sections and subsections.

Abstracts and Keywords

1. **Introduction**: in this section author briefly discuss meaning of the title and basic knowledge required.
2. **Overview**: in this section we can observe content of the paper.
3. **Related Work**: this section deals with the published work related to the main title.
4. **Methodology**: this section deals with explaining the method followed by authors for obtaining results.
5. **Main Result**: this section is the main section which carried new theorems, work, etc
6. **Applications**: this section deals with well-known problems and solution in short manner with apply introduced method.
   5.1. **Application** in obtaining formulae for $\int \boldsymbol{e^{ax}\cos bx\,dx}$ and $\int \boldsymbol{e^{ax}\sin bx\,dx}$
   5.2. **Application** in obtaining formula of $\boldsymbol{\mathcal{L}\{f^n(x)\}}$.
   5.3. **Application** in obtaining Toyesh's Generalizations.
   5.4. **Application** in obtaining Formulae for sine and cosine functions.
7. **Open Problem**: in this section there are some open problems for renders to found and further discussion.
8. **Conclusion**: this section deals with users of introduced method, further scope of the method, and summary of the work.

   Acknowledgement
   References
   Authors Profile

3. Related Work

An open problem asked in [4] and [5] was "Does there possible some other generalization for $\int e^x(f(x)+f'(x)dx$?" and "Does there possible some other generalization for $\int e^{-x}(f(x)-f'(x)dx$?". The answer to this problem was first given in [6].

Now, this paper will also deal with the solution to the problem and other than it the purpose of this paper combines both [1][2] which we can observe in application section respectively.

4. Methodology

The methodology followed for obtaining results contains a pack of various mathematical concepts like Mathematical Induction, Integration by parts for elementary functions, algebraic manipulation and substitution, observing pattern is very essential and on the basis of them in further sections author provide the main generalization and for further rechecking authors also proved some well known results from integral calculus and Laplace transform using their formula which is sufficient for saying the given main result is working nicely.

5. Main Result

We will look toward generalization by proving some new theorems and after it by using normal observation and concept of induction, stated main theorem respectively. So, we'll start with theorem 5.1.

Theorem 5.1.

$$\int e^{ax}(af(x) + f'(x)dx = e^{ax}f(x) + c$$

Proof:

Applying integration by parts in $\int e^{ax}f(x)dx$ then,

$$\int e^{ax}f(x)\,dx = f(x)\int e^{ax}dx - \int\left(\frac{df(x)}{dx}\int e^{ax}dx\right)dx$$

$$\Rightarrow \int e^{ax}f(x)\,dx = \frac{e^{ax}}{a}f(x) - \frac{1}{a}\int e^{ax}f'(x)dx \cdots\cdots\cdots (i)$$

$$\Rightarrow \int e^{ax}f(x)\,dx + \frac{1}{a}\int e^{ax}f'(x)\,dx = \frac{e^{ax}}{a}f(x) + c$$

$$\Rightarrow \int e^{ax}\left(f(x) + \frac{f'(x)}{a}\right)dx = \frac{e^{ax}}{a}f(x) + c$$

$$\Rightarrow \int e^{ax}(af(x) + f'(x)dx = e^{ax}f(x) + c$$

Hence prove.

Theorem 5.2.

$$\int e^{ax}\left(a^2f(x) - f''(x)\right)dx = ae^{ax}f(x) - e^{ax}f'(x) + c$$

Proof:

By applying integral by parts in (i),

$$\Rightarrow \int e^{ax}f(x)\,dx = \frac{e^{ax}}{a}f(x) - \frac{1}{a}\left(f'(x)\int e^{ax}dx - \int\left(\frac{df'(x)}{dx}\int e^{ax}dx\right)dx\right)$$

$$\Rightarrow \int e^{ax}f(x)\,dx = \frac{e^{ax}}{a}f(x) - \frac{1}{a}\left(\frac{e^{ax}}{a}f'(x) - \int\frac{e^{ax}}{a}f''(x)\,dx\right)dx$$

$$\Rightarrow \int e^{ax}f(x)\,dx = \frac{e^{ax}}{a}f(x) - \frac{e^{ax}}{a^2}f'(x) + \int\frac{e^{ax}}{a^2}f''(x)\,dx$$

Now, multiplying a^2 on both sides,

$$\int a^2e^{ax}f(x)dx = ae^{ax}f(x) - e^{ax}f'(x) + \int e^{ax}f''(x)dx \ldots\ldots\ldots (ii)$$

$$\Rightarrow \int e^{ax}\left(a^2f(x) - f''(x)\right)dx = ae^{ax}f(x) - e^{ax}f'(x) + c$$

Hence prove.

Theorem 5.3.

$\int e^{ax}\left(a^3f(x) + f'''(x)\right)dx = a^2e^{ax}f(x) - ae^{ax}f'(x) + e^{ax}f''(x) + c$

Proof:

By applying integral by parts in (ii),

$$\int a^2e^{ax}f(x)dx = ae^{ax}f(x) - e^{ax}f'(x) + \int e^{ax}f''(x)dx$$

$$\Rightarrow \int a^2e^{ax}f(x)\,dx = ae^{ax}f(x) - e^{ax}f'(x) + \left(f''(x)\int e^{ax}dx - \int\left(\frac{df''(x)}{dx}\int e^{ax}dx\right)dx\right) + c$$

$$\Rightarrow \int a^2e^{ax}f(x)\,dx = ae^{ax}f(x) - e^{ax}f'(x) + \left(\frac{e^{ax}}{a}f''(x) - \int f'''(x)\frac{e^{ax}}{a}\right)dx + c$$

Now, multiplying a on both sides,

$$\int a^3e^{ax}f(x)dx = a^2e^{ax}f(x) - ae^{ax}f'(x) + e^{ax}f''(x) - \int e^{ax}f'''(x)dx + c \quad \ldots\ldots.(iii)$$

$$\Rightarrow \int e^{ax}\left(a^3f(x) - f'''(x)\right)dx = a^2e^{ax}f(x) - ae^{ax}f'(x) + e^{ax}f''(x) + c$$

Hence prove.

Theorem 5.4.
$\int e^{ax}\left(a^4 f(x) - f''''(x)\right)dx = a^3 e^{ax} f(x) - a^2 e^{ax} f'(x) + a e^{ax} f''(x) - e^{ax} f'''(x) + c$

Proof:
By applying integral by parts in (iii),

$$\int a^3 e^{ax} f(x)dx = a^2 e^{ax} f(x) - a e^{ax} f'(x) + e^{ax} f''(x) - \int e^{ax} f'''(x)dx + c$$

$$\Rightarrow \int a^3 e^{ax} f(x)\,dx = a^2 e^{ax} f(x) - a e^{ax} f'(x) + e^{ax} f''(x) - \left(f'''(x)\int e^{ax}dx - \int\left(\frac{df'''(x)}{dx}\int e^{ax}dx\right)dx\right) + c$$

$$\Rightarrow \int a^3 e^{ax} f(x)\,dx = a^2 e^{ax} f(x) - a e^{ax} f'(x) + e^{ax} f''(x) - \left(f'''(x)\frac{e^{ax}}{a} - \int\frac{e^{ax}}{a} f''''(x)dx\right)dx + c$$

Now, multiplying a on both sides,

$$\int a^4 e^{ax} f(x)dx = a^3 e^{ax} f(x) - a^2 e^{ax} f'(x) + a e^{ax} f''(x) - f'''(x)e^{ax} + \int e^{ax} f''''(x)\,dx + c$$

$$\Rightarrow \int e^{ax}\left(a^4 f(x) - f''''(x)\right)dx = a^3 e^{ax} f(x) - a^2 e^{ax} f'(x) + a e^{ax} f''(x) - e^{ax} f'''(x) + c$$

Now by observing above theorems and by induction we are presenting main generalization i.e.

Generalization
If $n \in 1,2,3,4, \dots\dots\dots then,$

$$\int e^{ax}\left(a^n f(x) - (-1)^n \frac{d^n f(x)}{dx^n}\right)dx = e^{ax}\sum_{k=1}^{n} a^{n-k}(-1)^{k-1}\frac{d^{k-1} f(x)}{dx^{k-1}} + c$$

6. Applications

6.1 Application in obtaining formulae for $\int e^{ax}\cos bx\,dx$ and $\int e^{ax}\sin bx\,dx$

Theorem 1.

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$$

Proof
From the above proved theorem we have

$$\int e^{ax}\left(a^n f(x) - (-1)^n \frac{d^n f(x)}{dx^n}\right)dx = e^{ax}\sum_{k=1}^{n} a^{n-k}(-1)^{k-1}\frac{d^{k-1} f(x)}{dx^{k-1}} + c$$

Let $n = 2$ then,

$$\int e^{ax}\left(a^2 f(x) - \frac{d^2 f(x)}{dx^2}\right)dx = e^{ax}\left(af(x) - \frac{df(x)}{dx}\right) + c$$

Consider $f(x) = \cos bx$ then, $\frac{d}{dx}f(x) = -b\sin bx$ and $\frac{d^2}{dx^2}f(x) = -b^2\cos bx$ so,

$$\int e^{ax}\left(a^2\cos bx - (-b^2\cos bx)\right)dx = e^{ax}\left(a\cos bx - (-b\sin bx)\right) + c$$

$$(a^2 + b^2)\int e^{ax}\cos bx\,dx = e^{ax}(a\cos bx + b\sin bx) + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + \frac{c}{a^2 + b^2}$$

As a, b, c are constants then $\frac{c}{a^2+b^2}$ is also a constant that we can named as C, then

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$$

Theorem 2.

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$$

Proof
From the above proved theorem we have

$$\int e^{ax}\left(a^n f(x) - (-1)^n \frac{d^n f(x)}{dx^n}\right)dx = e^{ax}\sum_{k=1}^{n} a^{n-k}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}} + c$$

Let $n = 2$ then,

$$\int e^{ax}\left(a^2 f(x) - \frac{d^2 f(x)}{dx^2}\right)dx = e^{ax}\left(af(x) - \frac{df(x)}{dx}\right) + c$$

Consider $f(x) = \sin bx$ then, $\frac{d}{dx}f(x) = b\cos bx$ and $\frac{d^2}{dx^2}f(x) = -b^2\sin bx$ so,

$$\int e^{ax}\left(a^2\sin bx - (-b^2\sin bx)\right)dx = e^{ax}(a\sin bx - b\cos bx) + c$$

$$(a^2+b^2)\int e^{ax}\sin bx\, dx = e^{ax}(a\sin bx - b\cos bx) + c$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + \frac{c}{a^2+b^2}$$

As a, b, c are constants then $\frac{c}{a^2+b^2}$ is also a constant that we can named as C, then

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$$

6.2 Application in obtaining formula of $\mathcal{L}\{f^n(x)\}$.

This is a property of Laplace transform and one can see it in [7].

Theorem 3.

$$\mathcal{L}\{f^n(x)\}_p = p^n\mathcal{L}\{f(x)\}_p - p^{n-1}f(0) - p^{n-2}f'(0) - \cdots - pf^{n-2}(0) - f^{n-1}(0)$$

Proof

Considering $a = -p$ in the generalization we have obtain also putting lower and upper limits be zero and infinity then

$$\int_0^\infty e^{-px}\left((-1)^n p^n f(x) - (-1)^n\frac{d^n f(x)}{dx^n}\right)dx = \left[e^{-px}\sum_{k=1}^{n}(-p)^{n-k}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}\right]_0^\infty$$

$$\Rightarrow \int_0^\infty e^{-px}\left(p^n f(x) - \frac{d^n f(x)}{dx^n}\right)dx = -\left[e^{-px}\sum_{k=1}^{n}p^{n-k}\frac{d^{k-1}f(x)}{dx^{k-1}}\right]_0^\infty$$

$$\Rightarrow p^n\int_0^\infty e^{-px}f(x)\,dx - \int_0^\infty e^{-px}\frac{d^n f(x)}{dx^n}dx = -\left[e^{-px}\sum_{k=1}^{n}p^{n-k}\frac{d^{k-1}f(x)}{dx^{k-1}}\right]_0^\infty$$

From the definition of Laplace Transform we have $\mathcal{L}\{f(x)\}_p = \int_0^\infty e^{-px}f(x)dx$ then

$$p^n\mathcal{L}\{f(x)\}_p - \mathcal{L}\left\{\frac{d^n f(x)}{dx^n}\right\}_p = -\left[e^{-px}\sum_{k=1}^{n}p^{n-k}\frac{d^{k-1}f(x)}{dx^{k-1}}\right]_0^\infty$$

$$p^n\mathcal{L}\{f(x)\}_p - \mathcal{L}\left\{\frac{d^n f(x)}{dx^n}\right\}_p = -\lim_{x\to\infty} e^{-px}\sum_{k=1}^{n}p^{n-k}\frac{d^{k-1}f(x)}{dx^{k-1}} + \left(\sum_{k=1}^{n}p^{n-k}\frac{d^{k-1}f(x)}{dx^{k-1}}\right)_0$$

As $\lim\limits_{x\to\infty} e^{-px}\sum_{k=1}^{n}p^{n-k}\frac{d^{k-1}f(x)}{dx^{k-1}} = 0$ then

$$p^n\mathcal{L}\{f(x)\}_p - \mathcal{L}\left\{\frac{d^n f(x)}{dx^n}\right\}_p = p^{n-1}f(0) + p^{n-2}f'(0) + \cdots + pf^{n-2}(0) + f^{n-1}(0)$$

$$\Rightarrow \mathcal{L}\left\{\frac{d^n f(x)}{dx^n}\right\}_p = \mathcal{L}\{f^n(x)\} = p^n\mathcal{L}\{f(x)\}_p - p^{n-1}f(0) - p^{n-2}f'(0) - \cdots - pf^{n-2}(0) - f^{n-1}(0)$$

Hence prove

6.3 Application in obtaining Toyesh's Generalizations.

In 2020 and 2021 Toyesh gave his generalization as follows

Toyesh Generalization for $n \in \mathbb{N}$

$$\int e^x\left(f(x) - (-1)^n\frac{d^n f(x)}{dx^n}\right)dx = e^x\sum_{k=1}^{n}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}} + c$$

And

$$\int e^{-x}\left(f(x)-\frac{d^n f(x)}{dx^n}\right)dx=-e^{-x}\sum_{k=1}^{n}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

Proof

Putting $a=1$ in the obtained generalization above gives

$$\int e^{x}\left(f(x)-(-1)^n\frac{d^n f(x)}{dx^n}\right)dx=e^{x}\sum_{k=1}^{n}(1)^{n-k}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

$$\Rightarrow\int e^{x}\left(f(x)-(-1)^n\frac{d^n f(x)}{dx^n}\right)dx=e^{x}\sum_{k=1}^{n}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

This result was obtained by Toyesh Prakash Sharma [1][4] while on the other hand for $a=-1$ we can obtain

$$\int e^{-x}\left((-1)^n f(x)-(-1)^n\frac{d^n f(x)}{dx^n}\right)dx=e^{-x}\sum_{k=1}^{n}(-1)^{n-k}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

$$\Rightarrow(-1)^n\int e^{-x}\left(f(x)-\frac{d^n f(x)}{dx^n}\right)dx=e^{-x}\sum_{k=1}^{n}(-1)^{(n-1)-(k-1)}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

$$\Rightarrow(-1)^n\int e^{-x}\left(f(x)-\frac{d^n f(x)}{dx^n}\right)dx=e^{-x}(-1)^{(n-1)}\sum_{k=1}^{n}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

$$\Rightarrow\int e^{-x}\left(f(x)-\frac{d^n f(x)}{dx^n}\right)dx=-e^{-x}\sum_{k=1}^{n}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

This result was obtained by Toyesh Prakash Sharma [2][3].

6.4 Formulae for sine and cosine functions.

Theorem 4:

If $p=1,2,3,4\cdots$ then

$$\int\cos x\left\{f(x)-(-1)^p\frac{d^{2p}f(x)}{dx^{2p}}\right\}dx=v\cos x+u\sin x+C$$

And

$$\int\sin x\left\{f(x)-(-1)^p\frac{d^{2p}f(x)}{dx^{2p}}\right\}dx=v\sin x-u\cos x+C$$

Where, $u=\left(f(x)-f^2(x)\right)+\left(f^4(x)-f^6(x)\right)+\cdots$

And $v=\left(f^1(x)-f^3(x)\right)+\left(f^5(x)-f^7(x)\right)+\cdots$

Proof

As the proven generalization is

$$\int e^{ax}\left(a^n f(x)-(-1)^n\frac{d^n f(x)}{dx^n}\right)dx=e^{ax}\sum_{k=1}^{n}a^{n-k}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

For $a=-i$ and $n=2p$

we can obtain

$$\int e^{ix}\left(i^{2p}f(x)-(-1)^{2p}\frac{d^{2p}f(x)}{dx^{2p}}\right)dx=e^{ix}\sum_{k=1}^{2p}i^{2p-k}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

$$\Rightarrow\int e^{ix}\left((-1)^p f(x)-\frac{d^{2p}f(x)}{dx^{2p}}\right)dx=e^{ix}\sum_{k=1}^{2p}i^{2p}i^{-k}\frac{i^{-k}}{i^{-k}}(-1)^{k-1}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

$$\Rightarrow\int e^{ix}\left((-1)^p f(x)-\frac{d^{2p}f(x)}{dx^{2p}}\right)dx=e^{ix}\sum_{k=1}^{2p}(-1)^{p-k}(-1)^{k-1}i^{k}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

$$\Rightarrow\int e^{ix}\left((-1)^p f(x)-\frac{d^{2p}f(x)}{dx^{2p}}\right)dx=e^{ix}(-1)^{p-1}\sum_{k=1}^{2p}i^{k}\frac{d^{k-1}f(x)}{dx^{k-1}}+c$$

Using Euler's Identity $e^{ix}=\cos x+i\sin x$

then

$$\int (\cos x + i\sin x)\left((-1)^p f(x) - \frac{d^{2p} f(x)}{dx^{2p}}\right) dx = (-1)^{p-1}(\cos x + i\sin x)\sum_{k=1}^{2p} i^k \frac{d^{k-1} f(x)}{dx^{k-1}} + c$$

Now,

$$\sum_{k=1}^{2p} i^k \frac{d^{k-1} f(x)}{dx^{k-1}} = \begin{pmatrix} if(x) + i^2 f^1(x) + i^3 f^2(x) + i^4 f^3(x) + \\ i^5 f^4(x) + i^6 f^5(x) + i^7 f^6(x) + i^8 f^7(x) + \\ i^9 f^8(x) + i^{10} f^9(x) + i^{11} f^{10}(x) + i^{12} f^{13}(x) + \\ \dots\dots\dots \end{pmatrix}$$

$$= \begin{pmatrix} if(x) - f^1(x) - if^2(x) + f^3(x) + \\ if^4(x) - f^5(x) - if^6(x) + f^7(x) + \\ i(x) - f^9(x) - if^{10}(x) + f^{13}(x) + \\ \dots\dots\dots \end{pmatrix}$$

$$= \begin{matrix} i\{(f(x) - f^2(x)) + (f^4(x) - f^6(x)) + \cdots\} \\ -\{(f^1(x) - f^3(x)) + (f^5(x) - f^7(x)) + \cdots\} \end{matrix}$$

Consider $u = (f(x) - f^2(x)) + (f^4(x) - f^6(x)) + \cdots$ and $v = (f^1(x) - f^3(x)) + (f^5(x) - f^7(x)) + \cdots$

Then

$$\sum_{k=1}^{2p} i^k \frac{d^{k-1} f(x)}{dx^{k-1}} = -v + iu$$

Now,

$$\int (\cos x + i\sin x)\left((-1)^p f(x) - \frac{d^{2p} f(x)}{dx^{2p}}\right) dx = (-1)^{p-1}(\cos x + i\sin x)(-v + iu) + c$$

$$\Rightarrow \int \cos x\left\{(-1)^p f(x) - \frac{d^{2p} f(x)}{dx^{2p}}\right\} dx + i\int \sin x\left\{(-1)^p f(x) - \frac{d^{2p} f(x)}{dx^{2p}}\right\} dx$$
$$= (-1)^{p-1}[i(u\cos x - v\sin x) - (v\cos x + u\sin x)] + c$$

$$\Rightarrow \int \cos x\left\{(-1)^p f(x) - \frac{d^{2p} f(x)}{dx^{2p}}\right\} dx + i\int \sin x\left\{(-1)^p f(x) - \frac{d^{2p} f(x)}{dx^{2p}}\right\} dx$$
$$= (-1)^p(v\cos x + u\sin x) + i(-1)^{p-1}(u\cos x - v\sin x) + c$$

On comparing real part with real and imaginary part with imaginary we can say that

$$\int \cos x\left\{(-1)^p f(x) - \frac{d^{2p} f(x)}{dx^{2p}}\right\} dx = (-1)^p(v\cos x + u\sin x) + C$$

And

$$\int \sin x\left\{(-1)^p f(x) - \frac{d^{2p} f(x)}{dx^{2p}}\right\} dx = (-1)^p(v\sin x - u\cos x) + C$$

With multiplying $(-1)^p$ on both sides gives

Where, $u = (f(x) - f^2(x)) + (f^4(x) - f^6(x)) + \cdots$

and $v = (f^1(x) - f^3(x)) + (f^5(x) - f^7(x)) + \cdots$

$$\int \cos x\left\{f(x) - (-1)^p \frac{d^{2p} f(x)}{dx^{2p}}\right\} dx = (v\cos x + u\sin x) + C$$

And

$$\int \sin x\left\{f(x) - (-1)^p \frac{d^{2p} f(x)}{dx^{2p}}\right\} dx = v\sin x - u\cos x + C$$

Hence prove.

7. Open Problems

Some open problems are as follows: -

1. Prove the main generalization without using induction.

2. Does there possible some other generalization for
$$\int e^{ax}\left(a^n f(x) - (-1)^n \frac{d^n f(x)}{dx^n}\right) dx$$

3. Does main generalization can connect with Taylor series?
4. Like 5.4th application can we further introduce some other theorems for some other functions rest of sine and cosine function?

8. Conclusion and Future Scope

The purpose of the paper is to introduce another generalizations of $\int e^x(f(x) + f'(x)dx$ and $\int e^{-x}(f(x) - f'(x)dx$ other than the generalizations presented in [1][2] authors have successfully fulfill the purpose of the paper and provided new generalization after applying novel integration techniques such as substitution, Integration by parts and others. Further authors also provide applications of the main result that shows the versatility of the formula respectively. For further scope one can think about some other applications of the mail result, also one can pile the problems based on the given generalizations, one can thinks for giving answer to the above provided open problems.

Data Availability
None

Conflict of Interest
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Author-1, researched literature, Application and conceived the study.
Author-2, obtained main result and analyzing the generalizing.

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