Research Paper

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Approximation of D.I.Y. Water Rocket Dynamics Including Air Drag

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Abstract—If you want to get accurate predictions for the motion of water and air propelled D.I.Y rockets, neglecting air resistance is not an option. But the theoretical analysis including air drag leads to a system of differential equations which can only be solved numerically. We propose an approximation which simply works by the estimation of a definite integral, and which is even feasible for undergraduate physics courses. The results only slightly deviate from the reference data (received by the Runge-Kutta method). The motion is divided into several flight phases that are discussed separately and the resulting equations are solved by analytic and numeric methods. The different results from the flight phases are collected and are compared to data that has been achieved by well explained and documented experiments. Furthermore, we theoretically estimate the rockets drag coefficient. The result is confirmed by a wind tunnel experiment.

I. INTRODUCTION

In our paper, the ascent of the rocket is devided into water thrust phase, air thrust phase and upward coast phase. We also set up the equations of motion for the rocket in Section III. The equation that governs the gas expansion inside the rocket tank is solved analytically in Section IV, but it is not possible to analytically solve the complete system of differential equations that describe the rocket's launch. First and foremost, we study the water thrust phase in Section V. On the one hand, we apply the Runge-Kutta method to the corresponding initial value problem to have some proper reference data. On the other hand, we deduce a simple method to approximate the results with high accuracy: Our approximation simply works by the estimation of a definite integral. One advantage is that this calculation can be done with a simple graphing calculator without a computer algebra system and actually is feasible for undergraduate physics courses. Nonetheless, one receives very accurate results which only slightly deviate from the reference data (received by the Runge-Kutta method). Section VI is concerned with the air thrust phase. D.I.Y. rockets are chaotic systems and there is no way to receive analytic results. Therefore, we make reasonable assumptions to simplify the underlying equations and introduce an efficiency factor for the air thrust. In Section VII the upward coast phase is discussed. A collection of the previous results is given in Section VIII. Subsequently, our paper contains a detailed theoretical estimation for the drag coefficient and we verify the results by experimental data from a wind tunnel experiment, see Section IX. Finally, we compare our estimation of the rocket's

maximum altitude with the experimental data from our D.I.Y. rocket launch experiments. As these investigations show, our proposed method is suitable to predict the altitude of a D.I.Y rocket, which can be regarded as a highly chaotic system.

II. RELATED WORK

The classical rocket equation [15, eq. 2.12] is attributed to Konstantin Eduardovich Tsiolkovsky [23]. If the propellant is exhausted with constant speed the rocket equation can easily be integrated. In case of water and air propelled rockets the exhaust speed decreases together with the internal pressure and mass of the rocket. Physics of water rockets have been the subject of several investigations. In an early paper [17, p. 152], Nelson and Wilson claim that rocket thrust and mass as a function of time, as well as the drag coefficient must be determined experimentally. Later, Finney assumes in [9] that "*the air in the rocket behaves as an ideal gas and* [...] *expands isothermally*". He uses Bernoulli's equation to determine an equation for the mass flow rate of the propellant, see [9, eq. 3]. Furthermore, Finney deduces an equation for the pressure as a function of time and calculated the burn time of the rocket, see [9, eq. 7, 8]. Therewith, he proposes the estimation

$$h = \frac{1}{8}gt^2 - \frac{D}{64M}g^2t^4 \qquad (1)$$

for the height of the rocket, see [9, eq. 14], where $D = \frac{1}{2}\rho_{air}C_D A_R$ contains the parameters of the drag force F_D

and M is the mass of the empty rocket. The results were compared with a numerical analysis. As mentioned in the appendix of his paper "*an adiabatic approximation would seem more natural*." Gommes slightly improves the thrust prediction for water rockets: "*the gas expansion has to be modeled as an adiabatic process.*", cf. [10]. He also includes the fact that "*Air expansion [...] is accompanied by vapor condensation*". A more meticulous investigation of the thermodynamics of the water rocket's thrust phase was published by Romanelli, Bove and Madina in [19]. Indeed, the value of the polytropic exponent n in $pV^n = constant$ affects the rockets maximum altitude. There are several other effects that raise the inaccuracy of the predictions more than that. Prusa used $n = 1.4$ for dry air, see [18, p. 724]. In our paper we will take into account the enhanced value for n. Since n is a constant this does not cause additional difficulty. Now let us get back to the rockets movement: Prusa derived an equation for the acceleration of the rocket, cf. [18, eq. 2.2] and proposed a numerical algorithm to solve it, see [18, p. 723]. Prusa neglected the additional thrust from the compressed air which is left at the end of the water propulsion phase. An even more accurate consideration of the water rocket physics was given by Barrio-Perott et. al. in 2010, see [3]. They worked out a set of differential equations for the water thrust, cf. [3, eq. 18-21], as well as equations for the air thrust [3, eq. 28-31]. Both articles, [18] and [3], use numerical methods to achieve the solution.

III. THE ACCELERATION OF THE ROCKET

Consider a rocket moving with velocity v in vertical direction. Let m be the mass of the rocket including its propellant at a given time t. During the interval of time dt the rocket ejects the mass element dm with the exhaust speed v_e. According to the law of conservation of momentum the velocity v of the rocket thereby increases by $dv = -v_e m^{-1} dm$, cf. e.g. [15, eq. 2.11]. Consequently, the thrust acceleration is given by

$$a_I = -\frac{dm}{dt}\frac{v_e}{m} \qquad (2)$$

Since the total mass is shrinking, i.e. $\frac{dm}{dt} < 0$, it is $a_I > 0$. A rocket within the terrestrial atmosphere additionally experiences deceleration from gravity and air drag. Both of them decrease with the height of the rocket. However, a simple water rocket does only reach a maximum altitude of a few meters[1]. Consequently, the altitude dependence of gravity and air drag can be neglected. For the magnitude

1 It should be mentioned, that a group of scientists of the University of Cape Town built a water and air propelled rocket that reached 830m in 2015. This is the current record, cf. http://www.wra2.org/ and https://www.news.uct.ac.za/article/-2015-10-07-uct-team-smashes-eight-year-water-rocket-world-altitude-record. Certainly, a simple home-made rocket built from a 1 or 2 liter PET bottle is not even in a position to achieve this altitude.

$g = GM_{earth}r^{-2}$ of barycentric gravitational acceleration we use $g = 9.81ms^{-2}$ during the numerical calculations. Air resistance is modeld by the drag force $F_D = \frac{1}{2}\rho_{air}C_D A_R v^2$ where ρ air is the density of the air, c_D the drag coefficient, A_R the reference area (later we use the cross sectional area A_{cs} of the rocket perpendicular to the direction of movement), and v again the velocity of the rocket. Let us adopt the abbreviation

$$D = \frac{1}{2}\rho_{air}C_D A_R \qquad (3)$$

from [9]. The drag force $F_D = Dv^2$ leads to a deceleration

$$a_D = \frac{F_D}{m} = \frac{D}{m}v^2. \qquad (4)$$

An estimation for the drag coefficient of the model rocket is given in section VIII. Thrust acceleration and deceleration from gravity and drag point in the opposite direction. Therefore, the total acceleration of the rocket is given by

$$a = a_I - g - a_D = -\frac{v_e dm}{mdt} - g - \frac{D}{m}v^2. \qquad (5)$$

For a water rocket, the working mass (water) represents the major part of the rockets total mass m. Anyway, the total mass strongly decreases with time. In order to integrate (5) it is necessary to know the dependence of mass m and time t .

Mass and exhaust velocity

Let $m_w(t)$ be the time dependent working mass, namely the mass of the water in the rocket tank. Further let M denote the constant mass of the empty rocket. The time dependend mass of the compressed air does not contribute significantly to the total mass of the rocket. Despite this fact we will take it into account. Section VI is concerned with the additional air propulsion after the water thrust phase. During the water thrust phase we treat the mass of the compressed air as a constant, say m_{air}. This ansatz is based on the idea that all of the water is expelled before the air escapes. Of course this is somehow physically unrealistic. Indeed, an intermixture of water and air is expelled, especially towards the end of the water thrust phase. This yields a non-calculable chaos. Within our model assumption the mass of the rocket is given by $m(t) = M + m_{air} + m_w(t)$ during the water thrust phase. Let ρ denote the density of the working mass, in case of water this is $\rho \approx 1000kgm^{-3}$. $V_w(t)$ is the volume of the working mass. V_b is the volume of the rocket tank and $V(t) = V_b - V_w(t)$ the part which is filled with air (or another gas, maybe water vapor etc.). Accordingly, the mass m of the rocket and the differential dm read

$$m = M + m_{air} + \rho\big(V_b - V(t)\big) \Rightarrow dm = -\rho dV. \qquad (6)$$

Thereby, the dependence of mass m and time t is determined by the expansion of the gas during the water thrust phase. Let p a denote the atmospheric pressure and $p = p(V)$ the pressure of propellant and gas inside the rocket. Initial values (at rocket launch) for gas pressure and volume are denoted by p_0 and V_0, respectively. Bernoullis equation $p = p_a + v_e^2 \rho/2$, see [22], relates the exhaust speed v_e to the pressure difference $p - p_a$:

$$v_e = \sqrt{\frac{2p(V) - p_a}{\rho}}. \qquad (7)$$

Indeed, the barometric pressure p a slightly depends on the altitude, see [12]. At altitudes that can be reached by a water rocket, p a decreases by roughly 12 Pa/m . But since the pressure of the propellant will usually be in the range of a few bar ($1bar = 10^5 Pa$) we can neglect the altitude dependence of the barometric pressure. Another point is that the fluid's velocity before leaving the nozzle is assumed to be zero in (7). Strictly speaking, we have to take into account the rejuvenation of the bottle. Consider an incompressible fluid (like water) which flows (laminar) from a point inside the bottle with cross-section A_R through the nozzle with cross-section A at velocity w_e. Let R and r be the radii of bottle and nozzle, respectively. Usually, the radius of the nozzle will be small against the rocket's radius. The velovity of the fluid inside the bottle is determined by the continuity equation. From Bernoullis equation $p + (w_e r^2/R^2)^2 \rho/2 = p_a + w_e^2 \rho/2$ we receive the slightly more accurate exhaust velocity

$$w_e = \sqrt{\frac{2p(V) - p_a}{\rho}} \; \frac{1}{\sqrt{1 - \left(\frac{r}{R}\right)^4}}$$
$$= v_e \left(1 + \frac{r^4}{2R^4} + O\left(\left[\frac{r}{R}\right]^8\right)\right) \qquad (8)$$

As shown above equation (7) represents the 3rd taylor order approximation in r/R for the exhaust velocity w_e. But what is the order of magnitude of the error while using (7) instead of the latter equation? The radius of a commercially available PET bottle can be estimated to be about 4 to 5 cm . Our nozzle has a diameter of about 9 mm . That leads to $R \approx 10r$ and we get $w_e \approx 1.00005 v_e$. The error for the exhaust velocity is in the range of 0.005% . Even for a nozzle with r=R/3, which seems to be unrealistic large, the error will be less than 1% . Obviously, the approximation (7) is good enough for our concerns. The working mass pressure and therewith the pressure difference depend on the gas volume V. The propellant is expelled through a nozzle with a cross sectional area A at speed v_e. Let $d\vec{A}$ be the vector surface element normal to A. In our case, the velocity vector $\vec{v_e}$ is perpendicular to A as well. Accordingly, the volumetric flow rate through the plane surface A reduces to

$$\frac{dV}{dt} = \iint_A \langle \vec{v_e}, d\vec{A} \rangle = A v_e \qquad (9)$$

and from (6) we deduce the mass flow rate

$$\frac{dm}{dt} = \frac{d}{dt}\big[M + m_{air} + \rho(V_b - V(t))\big] = -\rho \frac{dV}{dt}$$
$$= -\rho A v_e \qquad (10)$$

Together with (7) and the above considerations the rockets acceleration (5) takes the form

$$a = \frac{\rho A v_e^2}{m} - g - \frac{D}{m} v^2$$
$$= \frac{2A(p(V) - p_a) - Dv^2}{M + m_{air} + \rho(V_b - V)} - g \qquad (11)$$

where V is the gas volume as a function of time t. In order to solve the latter equation it is also necessary to know the relation p(V) of pressure and volume.

IV. EXPANSION OF THE GAS DURING THE THRUST PHASE

"*Many real processes undergone by gases or vapours are approximately polytropic with a polytropic index typically between 1.0 and 1.7 [...]*", cf. [8]. It is argued in [10] that the gas expansion in a model rocket can also be desribed by a polytropic process. Pressure p and Volume V are related by $pV^n = constant$. If p_0 and V_0 denote the corresponding initial values this is

$$p(V) = p_0 \left(\frac{V_0}{V}\right)^n. \qquad (12)$$

Volumetric flow rate (9), exhaust speed (7) and (12) lead to

$$\frac{dV}{dt} = A v_e = A \sqrt{\frac{p_0 V_0^n V^{-n} - p_a}{\rho}}. \qquad (13)$$

In case of $V = V_b$ we receive the rocket's burn time by numerical integration of

$$t_b = \frac{1}{A}\sqrt{\frac{\rho}{2}} \int_{V_0}^{V_b} \frac{dV}{\sqrt{p_0 V_0^n V^{-n} - p_a}}. \qquad (14)$$

Actually, there is still left some compressed gas (air) after the whole water is expelled. This provides an additional air thrust. Strictly speaking, equation (14) gives the duration of the water-thrust phase. However, the duration of the air thrust phase is very short. We will discuss the air boost in Section VI.

     3

Polytropic index

Specific investigations of the gas expansion require the value of the polytropic exponent. Frequently, the polytropic index is choosen as n=1.4, see e.g. [14, 18]. This corresponds to the assumption that the gas in the rocket is dry air. As mentioned in [10] "*the air expansion in the rocket is accompanied by water vapor condensation, which provides an extra thrust*". The water vapour pressure at which water vapour is in thermodynamic equilibrium with the water depends on the temperature. For moist air the relation of saturation vapor pressure and temperature can be well approximated by an equation given by Arden Buck in [5]. If T denotes the air temperature in , the saturated vapour pressure p_v is given by

$$p_v = K exp\left[\left(18.678 - \frac{T}{234.5}\right)\left(\frac{T}{257.14 + T}\right)\right] \quad (15)$$

where $K = 611.21 Nm^{-2} = 6.1121 * 10^{-3} bar$. An approximation of the polytropic index that depends on the water vapor pressure is given by

$$n = 1.15 + (1.4 - 1.15)exp\left(-36\frac{p_v}{p_0}\right), \quad (16)$$

see [10] eq. 6 or [19] eq. 19. For example, $p0 = 3bar$ and $T = 15^o C$ yield $n \approx 1.35$. Figure 1 and 2 show the polytropic index in dependence of temperature and pressure, i.e. the function $n(T, p_0)$ given by (16) together with (15). For the expansion of moist air in water rockets the polytropic index is in the range $1.1 \leq n \leq 1.4$.
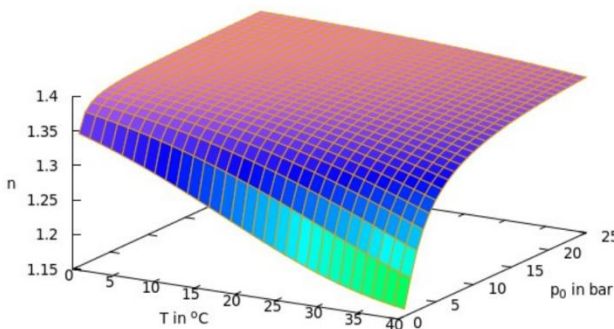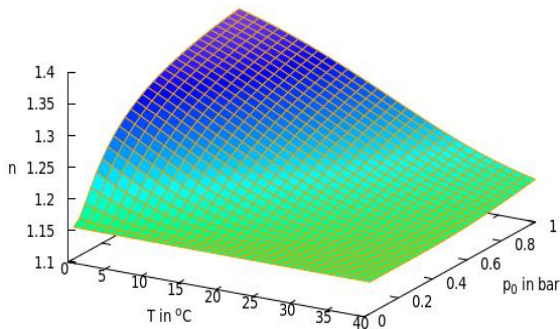

Figure 1. Polytropic index $n(T, p_0)$
where temperature ranges from $0^o C$ to $40^o C$.
The pressure is in the range of $0 \leq p_0 \leq 1bar$
and in the range of $0 \leq p_0 \leq 25bar$. (second grafic)

Analytic solution of the gas expansion equation

First, the question arises if there is an analytical solution for (13). If that is not the case, the corresponding equation can be solved numerically. The following approach is based on the Gaussian hypergeometric function. A little manipulation of the ordinary differential equation (13) for the volume yields

$$\frac{d}{dt}\left[\left(\frac{p_a}{p_0 V_0^n}\right)^{\frac{n+2}{2n}} V^{\frac{n+2}{2}}\right] = \frac{n+2}{2}\left(\frac{p_a}{p_0 V_0^n}\right)^{\frac{n+2}{2n}} \sqrt{\frac{2p_0 V_0^n}{\rho}}$$

$$\times A \sqrt{1 - \left[\left(\frac{p_a}{p_0 V_0^n}\right)^{\frac{n+2}{2n}} V^{\frac{n+2}{2}}\right]^{\frac{2n}{n+2}}}. \quad (17)$$

We define

$$u := \left(\frac{p_a}{p_0 V_0^n}\right)^{\frac{n+2}{2n}} V^{\frac{n+2}{2}}, \alpha := \frac{2n}{n+2},$$

$$k := \frac{n+2}{2}\left(\frac{p_a}{p_0 V_0^n}\right)^{\frac{n+2}{2n}} \sqrt{\frac{2p_0 V_0^n}{\rho}} A \quad (18)$$

and finally get

$$\frac{du}{\sqrt{1 - u^\alpha}} = kdt. \quad (19)$$

This equation admits the solution

$$uH\left(\frac{1}{2}, \frac{1}{\alpha}; \frac{1}{\alpha} + 1; u^\alpha\right) = kt + \xi0 \quad (20)$$

where and $u_0 = u(0)$.

Here $H(a, b; c; x) = \sum_{i=0}^{\infty} \frac{\Gamma(a+i)\Gamma(b+i)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+i)i!} x^i$ is a special generalized hypergeometric function aka Gauss hypergeometric function. More precisely, this function is one solution of Euler's hypergeometric differential equation, see [4, eq. 1.498],

$$x(x-1)w'' + ((a + b + 1)x - c)w' + abw = 0 \quad (21)$$

which can also be written as

$$\left(x\frac{d}{dx} + a\right)\left(x\frac{d}{dx} + b\right)w = \left(x\frac{d}{dx} + c\right)w'. \quad (22)$$

The Function H(a,b;c;x) obeys

$$\frac{d}{dx}\left(x^{c-1}H(a, b; c; x)\right) = (c-1)x^{c-2}H(a, b; c-1; x) \quad (23)$$

$$H(a, b; b; x) = (1-x)^{-a}, \qquad (24)$$

see [21, eq. 1.4.1.6] and [21, eq. 1.5.1]. Therefore, writing

$$g(x) = x^{\frac{1}{\alpha}} H\left(\frac{1}{2}, \frac{1}{\alpha}; \frac{1}{\alpha} + 1; x\right)$$

$$\mathcal{g}(u) = g(x(u)) = u H\left(\frac{1}{2}, \frac{1}{\alpha}; \frac{1}{\alpha} + 1; u^{\alpha}\right) \qquad (25)$$

with $x(u) = u^{\alpha}$, we get

$$\frac{d\mathcal{g}}{du} = \frac{1}{\sqrt{1 - u^{\alpha}}} \qquad (26)$$

as stated. Although a solution of (13) is represented by equation (20), its implicit form is inexpedient for our concerns. Of course, the burn time t_b can be calculated from (20). But we also need a closed-form expression for the gas volume as a function of time. Hence, the expansion equation (13) will be included into the system of equations of motion for our rocket. However, let us take a look at the numerical solution of (13) now. Within our calculations with the Runge Kutta method the temperature of water and vapor is chosen to be $15^o C$. The volume of the bottle and the initial water volume are $1 dm^3$ and $0.35 dm^3$, respectively. The corresponding Figure 2 was created with wxMaxima. The source code is available at https://github.com/tguent/code.
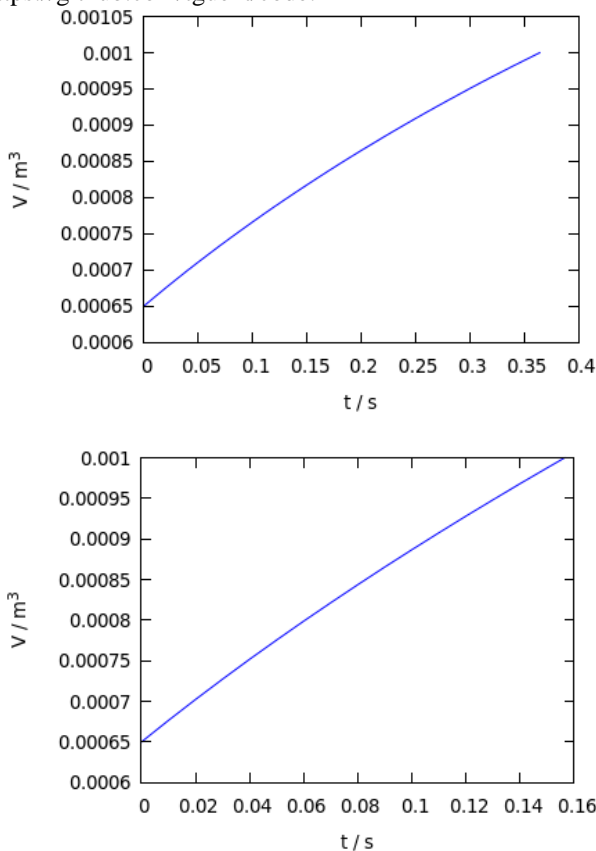

Figure 2. Expansion of water vapor at $15^o C$. The upper graphic shows the expansion with an initial pressure of 3 bar . This leads to a polytropic exponent of n=1.35. In the other case, the initial pressure is 10 bar, which leads to n=1.39.

V. MOVEMENT IN THE WATER THRUST PHASE

The flight of the rocket can be divided into three parts: Thrust phase, coast phase with upwards motion, and free-fall phase. We model the thrust phase by dividing it again into two parts: The main thrust is provided by the expelled water. During the water thrust phase the acceleration of the water rocket is modeled by (5). But usually there is still some compressed air left at the end of the water thrust phase. This gives rise to an additional nonzero air thrust. Accordingly, we refer to this as the air thrust phase in the following. After all of the propellant (water and air) is exhausted, the rocket can be regarded as an object that is thrown vertically upwards. The corresponding initial velocity is determined by the velocity at the end of the thrust phase. After reaching the maximum altitude, the rocket enters the free-fall phase. During the upward coast and free-fall phase air drag has a crucial influence on the rockets movement. As we will see in the following, air drag is negligible during the thrust phase of a water rocket.

Remark 1. **Data of reference example** For the numerical calculations we use the following data: The empty rocket has a mass of 1/8 kg and a volume of $1 dm^3$, its nozzle has a diameter of 9 mm. The rockets radius is 4 cm. Initial values for propellant volume and pressure are $V_0 = 0.35 dm^3$ and $p0 = 3^-$ at a temperature of $15^o C$. The mass of the compressed air amounts to roughly 1.8 g. The air drag coefficient was set to $c_D = 1$.

Water thrust phase including air drag

As mentioned in section IV, pressure p and volume V are related by polytropic expansion (12) where V as a function of time t is determined by equation (13). The velocity of the rocket can be expressed as the rate of change of its position h by $v = \frac{dh}{dt}$. Analogously, the acceleration (11) can be expressed as the rate of change of its velocity $a = \frac{dv}{dt}$. Finally, this leads to the following first order system of differential equations for gas volume V, altitude h and velocity v :

$$\frac{dV}{dt} = A\sqrt{\frac{p_0 V_0^n V^{-n} - p_a}{\rho}}, \frac{dh}{dt} = v,$$

$$\frac{dv}{dt} = \frac{2A(p_0 V_0^n V^{-n} - p_a) - Dv^2}{M + m_{air} + \rho(V_b - V)} - g \qquad (27)$$

The latter can be solved numerically. We use the fourth order Runge Kutta method, see [1], which is implemented in the computer algebra system wxMaxima [24]. The step-size was set to 0.01. The corresponding numerical results for the reference data given in Remark 1 are presented in Figure 3. In order to create comparability we use the following reference values $v_b \approx 15.32 ms^{-1}$ at an altitude of $h_b \approx 2.71 m$ at the end of the water thrust phase. For the
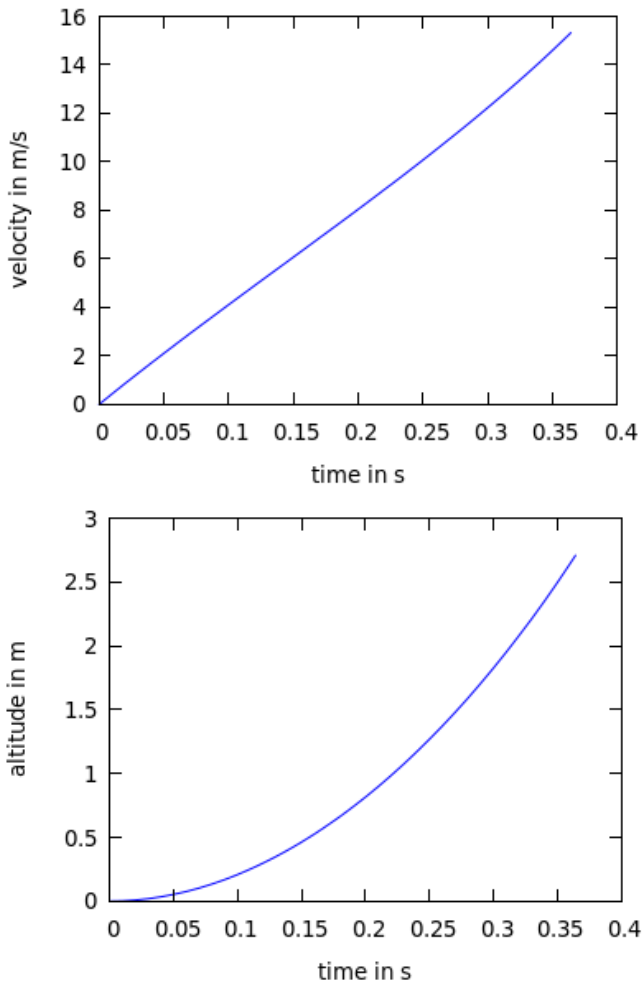
corresponding wxMaxima source code see again https://github.com/tguent/code.

Figure 3. Velocity and altitude of the rocket during the water thurst phase for the referce data, see Remark 1.

At the end of the water thurst phase the rocket has a velocity of $v_b \approx 15.32 ms^{-1}$ at an altitude of $h_b \approx 2.71m$.

Simple estimation - Water thrust phase approximation with zero drag coefficient

Solving the equations of motion (27) require advanced numerical calculations. In the following we present a more simple estimation for altitude and velocity at the end of the water thrust phase. The first equation of (27) determines the gas expansion. Figure 2 shows the gas expansion for $0 \le t \le t_b$, where t_b is determined by (14). This indicates that the gas volume as a function of time is not too far from being linear, at least as a rough approximation. If we use

$$V(t) = \frac{V_b - V_0}{t_b} t + V_0 \qquad (28)$$

for our estimation the corresponding equation of motion predicts a velocity of roughly $15.17 ms^{-1}$ at an altitude of 2.70 m at the end of the thrust phase (again for the data given in Remark 1). This is only a minor deviation

downwards from the more accurate reference values of $v_b \approx 15.32 ms^{-1}$ and $h_b \approx 2.71m$. The relative deviation of velocity and altitude at the end of the thrust phase lies in the range of about 0.9% and 0.2% , repectively. Thus, the linear ansatz (28) can be used as a good approximation. Now consider the last equation of (27) which incorporates thrust and air resistance. Although air drag has cruical influence on the maximum altitude of the rocket it has few influence during the thrust phase. During the subsequent free flight phase when the propellant is exhausted it is very important to incorporate air drag again. Next we estimate the data at the end of the water thrust phase in case of $c_D = 0$. Together with (28), equation (27) leads to

$$v_n(t) = \int_0^t \frac{2A\left[\frac{p_0 V_0^n}{\left(\frac{V_b - V_0}{t_b}x + V_0\right)^n} - p_a\right]}{M + m_{air} + \rho V_b - \rho\left(\frac{V_b - V_0}{t_b}x + V_0\right)} dx - gt \qquad (29)$$

at a given time $t \le t_b$. Integration over the burn time t_b, see (14), gives the velocity v_{nb} at the end of the water thrust phase. The altitude at the end of a water rockets thurst phase is small. Accordingly, we use the mean acceleration $\bar{a} = v_{nb}/t_b$ and

$$h_{nb} := \frac{1}{2}\bar{a}t_b^2 = \frac{v_{nb}t_b}{2} \qquad (30)$$

as a rough approximation for the altitude at the end of the water thurst phase. With the above used data from Remark1, this leads to velocity $v_{nb} \approx 15.60 ms^{-1}$ at an altitude $h_{nb} \approx 2.84m$ at the end of the water thurst phase. The results exceed the reference values of $v_b \approx 15.32 ms^{-1}$ and $h_b \approx 2.71m$ by 1.8% and 4,9% , repectively. Such a simple analysis of the motion of a water rocket can actually be carried out in undergraduate physics courses. All necessary calculations for the method figured out can be done with a graphing calculator.

VI. THE AIR THRUST PHASE

As mentioned before, the mass of air inside the rocket is comparatively small. Even if we neglect the mass of the compressed air during the water thrust phase the error will be very small. Using (12) the mass of air inside the rocket can be calculated from

$$m_{air} = \rho_{air} \sqrt[n]{\frac{p_0}{p_a}} V_0 \qquad (31)$$

where $\rho_{air} \approx 1.23 g dm^{-3}$ is the density of air at atmospheric pressure $p_a \approx 1bar$. For example: An initial

air volume of $V_0 = 0.65 dm^3$ at $p_0 = 3 bar$ contributes to the total mass of the rocket with about 2 g . In our case, the rocket has a curb weight of about 1/8 kg plus 350 g propellant (water). The additional mass of 2 g leads to a deviation that is in the range of some centimeters of calculated altitude. The additional mass of the compressed air does not has a noticeable influence on the water thrust phase. But what effect has the expelled air after the water thrust phase? On the one hand, there is only a tiny working mass left to induce a change in momentum. On the other hand, the air escapes very rapidly. Indeed, it already was a matter of discussion if the air boost can be neglected or not. Prusa notes: "*Due to its low density, the thrust provided by air alone is negligible, and in a launch without water, the rocket is barely able to lift off of the air pump seal.*", cf. [18, p. 719]. On the other hand, Barrio-Perotti et. al. state: "*This air is expelled through the nozzle causing an additional increase in the rocket momentum that sometimes cannot be neglected: for example, a rocket with air pressurized at 2 bars can reach a distance of about 10 m [...]*", cf. [3, p. 1138]. In [3], the air propulsion is also described by a system of differential equations, see [3, eq. 29, 30, 31] together with the algebraic equation [3, eq. 28] which have to be solved numerically. In order to decide whether the air boost has to be taken into account or not we filled our model rocket with compressed air (no water) at about 2 bar pressure. Actually, the rocket lifted off and reached a significant altitude of at least two meters. Thus we decided not to neglect the air propulsion. But since the effect is small compared to the water thrust, we choose a straightforward approximation for the air thrust phase in the following.

Rocket equation with simplified model assumptions

The gas volume equals the bottle's volume V_b now. According to (12), initial pressure p_0 and initial air density have reduced to

$$p_b = p_0 \left(\frac{V_0}{V_b}\right)^n \wedge \rho_b = \frac{m_{air}}{V_b} = \rho_{air} \frac{V_0}{V_b} \sqrt[n]{\frac{p_0}{p_a}} \qquad (32)$$

The change in momentum can be calculated from the rocket equation (5). In order to avoid complicated numerical methods we neglect the air drag during the air thrust phase. Barrio-Perotti et. al. evaluate water and air thrust in [3]. In comparison with the duration of the water thrust phase, the air thrust phase is short. Beyond that, the rockets acceleration is strongly decreasing during the air propulsion, see [3, Fig. 12]: If the air propulsion lasts about 0.025 seconds, the major contribution may be over within a hundredth of a second. Compared with this, the water thrust in [3, Fig. 12] lasts six times longer (and leads to higher acceleration anyway). Based on the short duration of the air propulsion, we assume that the exhaust velocity can be treated as a constant. With these simplifications the rocket equation $\frac{dv}{dt} = -\frac{v_e}{m}\frac{dm}{dt} - g$ can be integrated. It is

$$\int_{v_b}^{v_{max}} dv = -v_e \int_{M+M_{air}}^{M} \frac{dm}{m} - \int_d g \, dt \qquad (33)$$

where d is the duration of the air propulsion. Now let us rate the two terms on the right side of (33). Because the change of mass can be huge the integral over the mass may have a significant contribution to the result, despite of the fact that the duration of the air propulsion is very short. Now consider the last integral. The product of gravitational acceleration and duration of air propulsion will be also small. So we neglect this part. The remaining terms result in

$$v_{max} \approx v_e ln\left(\frac{M + m_{air}}{M}\right) + v_b \qquad (34)$$

where v_b is the rockets velocity at the end of the water thrust phase. Indeed, the above considerations are further based on the assumption that no air escapes before all of the water is expelled. This seems to be unrealistic. Particularly towards the end of the water thrust phase the propellant will be a mixture of water and air. Accordingly, the mass of the remaining air at the end of water thrust will be less than the initial amount of air inside the rocket given by (31). In order to take into account the loss of air during the water thrust phase we include an efficiency factor η and receive

$$v_{max} = v_e ln\left(\frac{M + \eta m_{air}}{M}\right) + v_b. \qquad (35)$$

Amongst other things, the efficiency factor η presumably depends on the initial proportion of water and air in the rocket, that is V_0/V_b. If the initial amount of air equals the bottle volume, there is no water propulsion and the air provides the whole thrust. Obviously the air thrust efficiency should be η=1 in this case. If the whole bottle is filled with water, η=0 should indicate that there will be no air thrust. Furthermore, the efficiency strongly depends on the construction of the D.I.Y rocket. Since we consider a highly chaotic system, there should be a parameter to readjust our calculation to the experimental data of a special model rocket. Based on the preceding considerations we choose

$$\eta = (V_0/V_b)^\mu \qquad (36)$$

As a first approximation one can use μ=1 . In order to fine-tune the estimation for a special D.I.Y. rocket, the exponent μ ≥ 0 can be determined from the experimantal data: For our model rocket a value of $\mu \approx 3$ works quite well.

Now we still need the exhaust velocity v_e. We can not simply calculate v_e from (7), since it is based on Bernoulli's equation for incompressible flow. The density

of a compressible flow is not constant. But the derivation of Bernoullis equation from $dp\rho^{-1} + vdv + gdz = 0$ can be modified for compressible flow. With $p \propto \rho^n$, see [7, eq. 3.20], we receive the relationship

$$\frac{n}{n-1} \cdot \frac{p}{\rho} + \frac{1}{2}v^2 = constant \tag{37}$$

which applies to compressible adiabatic flows, see [7, eq. 3.21]. We use again the above notation ρ_{air} for the air density at atmospheric pressure. Density and pressure at the beginning of the air thrust phase are denoted by an index b. Let us assume that the speed inside the bottle can be neglected. Therewith, we receive for the exhaust velocity:

$$v_e = \sqrt{\frac{2n}{n-1}\left(\frac{p_b}{\rho_b} - \frac{p_a}{\rho_{air}}\right)}$$
$$= \sqrt{\frac{2np_a}{(n-1)\rho_{air}}\left[\left(\frac{p_0}{p_a}\right)^{1-\frac{1}{n}}\left(\frac{V_0}{V_b}\right)^{n-1} - 1\right]} \tag{38}$$

Combining equations (35) and (38) results in

$$v_{max} \approx \sqrt{\frac{2np_a}{(n-1)\rho_{air}}\left[\left(\frac{p_0}{p_a}\right)^{1-\frac{1}{n}}\left(\frac{V_0}{V_b}\right)^{n-1} - 1\right]}$$
$$\times ln\left(\frac{M + \eta m_{air}}{M}\right) + v_b \tag{39}$$

Since the duration of the air propulsion is very short, we neglect the additional height which the rocket attains during this phase. But we take into account the enhanced velocity from the air thrust. This significantly affects the upward coast.

VII. MOVEMENT IN THE COAST PHASE

By the time the working mass is expelled the rocket has already reached its top speed. From this point we can regard the rocket as an object that is thrown vertically upwards. Air resistance has a significant impact on the movement after the thrust phase. Luckily, the corresponding equations of motion can be solved analytically. The kinematics of an upward movement including air resistance are well known. As the propellant of the rocket is exhausted the remaining mass M of the rocket is constant. With the notation

$$\psi = \frac{D}{M} = \frac{\rho_{air}c_D A_R}{2M} \tag{40}$$

the air drag deceleration (4) takes the form $a_D = \psi v^2$ during the free flight phase. For $v \leq v_{max}$ the equation of motion

$$\frac{dv}{dt} = -(g + \psi v^2) \tag{41}$$

can be integrated with the solution

$$v(t) = \sqrt{\frac{g}{\psi}}\,tan\left(arctan\left[\sqrt{\frac{\psi}{g}}v_{max}\right] - \sqrt{\psi g}t\right). \tag{42}$$

With v=0 the duration of the upward coast phase is given by

$$t_c = \frac{1}{\sqrt{\psi g}}\,arctan\left[\sqrt{\frac{\psi}{g}}v_{max}\right] \tag{43}$$

The upward coast begins after the thurst phase and ends once the rocket has reached the maximum altitude. The covered distance during the free flight phase is given by

$$h_c = -\frac{1}{\psi}ln\left|cos\left(arctan\left[\sqrt{\frac{\psi}{g}}v_{max}\right]\right)\right|. \tag{44}$$

Using $cos(arctanx) = 1/\sqrt{1 + x^2}$ one finally gets

$$h_c = \frac{1}{2\psi}ln\left(1 + \frac{\psi}{g}v_{max}^2\right) \tag{45}$$

VIII. COLLECTION OF THE RESULTS AND MAXIMUM ALTITUDE

Let us summarize the previous results and therewith derive a method to estimate the maximum altitude of the rocket. First we have to calculate the burn time of the water thrust phase by

$$t_b = \frac{1}{A}\sqrt{\frac{\rho}{2}}\int_{V_0}^{V_b}\frac{dV}{\sqrt{p_0V_0^nV^{-n} - p_a}}. \tag{46}$$

Here A is the nozzle cross sectional area, ρ the water density, n the polytropic exponent, p_a the atmospheric pressure and p_0 the initial pressure in the rocket tank. The tank has the volume V_b. At launch, $V_0 < V_b$ is filled with air and $V_b - V_0$ is the initial water volume. The water thrust phase can be modeled by the system of differential equations (27). Its numerical solution gives velocity v_b and altitude h_b at the end of the water thrust phase. Alternatively, for a rough estimation, we receive these values from

$$v_b \approx v_{nb} = \int_0^{t_b}\frac{2A\left[\frac{p_0V_0^n}{\left(\frac{V_b - V_0}{t_b}x + V_0\right)^n} - p_a\right]}{M + m_{air} + \rho V_b - \rho\left(\frac{V_b - V_0}{t_b}x + V_0\right)}dx - gt_b$$
$$h_b \approx h_{nb} = \frac{v_b t_b}{2}. \tag{47}$$

M is the curb mass of the rocket, m_{air} the initial mass of the air in the tank and g the gravitational acceleration. After the water thrust phase, the rocket experiences an additional acceleration due to the air propulsion. We only consider the enhanced velocity (39) within our model

$$v_{max} \approx \sqrt{\frac{2np_a}{(n-1)\rho_{air}}\left[\left(\frac{p_0}{p_a}\right)^{1-\frac{1}{n}}\left(\frac{V_0}{V_b}\right)^{n-1} - 1\right]} \\ \times ln\left(\frac{M+\eta m_{air}}{M}\right) + v_b \qquad (48)$$

where η is the efficiency factor of the air propulsion, see equation (36). The rocket enters the upward coast phase with the velocity v_{max} at the altitude h_b. During the upward coast the rocket reaches an additional height of

$$h_c = \frac{M}{\rho_{air}c_D A_R} ln\left(1 + \frac{\rho_{air}c_D A_R}{2Mg}v_{max}^2\right) \qquad (49)$$

see (45). Finally, the maximum altitude of the rocket can be estimated from $h_{max} = h_b + h_c$.

IX. DRAG ANALYSIS - AN ESTIMATION OF THE DRAG COEFFICIENT

The drag coefficient can be divided into its components which arise from pressure drag and friction drag, see [25, eq. 7.63]. Friction drag is caused by the viscosity of the surrounding air in our case. Pressure drag is the "*difference between the high pressure in the front stagnation region and the low pressure in the rear separated region* [...]", cf. [25, p. 448]. Basically, the drag coefficient varies with the Reynolds number Re=vL/ν where v is the velocity of the rocket, L is its characteristic length, and ν is the kinematic viscosity[2] of the surrounding atmosphere, see [25, eq. 7.61]. A simple home-made rocket built from a PET bottle achieves a maximum speed of about $20 ms^{-1}$. It is easy to estimate from Re=vL/ν that the Reynolds number will not exceed roughly $5 \cdot 10^5$ in this case. Of course, a more professional constructed water rocket might receive a maximum Reynolds number which is about ten times higher. The relation[3] of drag coefficient and Reynolds number is usually obtained from laboratory experiments, see for example [25, fig. 7.16]. Since such precise aerodynamic considerations are beyond the scope of this paper we try to estimate a constant drag coefficient for our rocket in the following. "*The drag analysis of rockets* [...] *is usually simplified by considering the rocket to be made up of several simple basic components*" [11]. We will

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2 The kinematic viscosity of air at 15^oC is about $1,5 \cdot 10^{-5}m^2s^{-1}$. The NASA provides a "Similarity Parameter Calculator" which calculates the Reynolds number: https://www.grc.nasa.gov/www/k-12/airplane/viscosity.html

3 It is worth mentioning that the drag force increases with increasing speed of the rocket. Although if the drag coefficient usually decreases.

confine our considerations to the drag analysis of nose cone, body tube, base, fins, and a (small) constant value for the launch lugs. The latter segmentation of a model rocket is shown in [11, fig. 17]. Due to interference drag the total drag of the rocket amounts to more than the sum of the components: "[...] *additional amount of drag is caused by the joining of the fins to the rocket body.* [...] *Interference drag can be as much as 10% above the sum of the fin and body tube drag*", cf. [11, p. 9]. In this paper, the drag coefficients of nose cone, body tube, base, fins, interference (int) and launch lugs (lau) are denoted by the names of the corresponding objects. The total drag coefficient c_D = nose + tube + base + fin + int + lau represents the case that the rocket is moving at zero angle to the direction of the wind. Any nonzero angle leads to an additional induced drag.

Nose cone and body tube of the rocket

The nose cone is exposed to presure drag and skin friction drag. As mentioned in [11, p. 10] a flat nose cone (solely) would result in a drag coefficient of 0.8 due to pressure. Gregorek moreover compares the latter case to the order of magnitude of drag coeffients of several shapes. Figure 23 in [11] indicates that rounding the nose reduces the corresponding drag coefficient by about 90% or even more. Nose cone and body tube of the rocket are additionally exposed to friction drag. Luckily, there exists an equation, see [11, eq. 8], which includes the drag of the nose cone as well as the body tube. Let A_{cs} be the cross-sectional area of the body tube and A_{ws} the wetted surface area of the rocket. Therewith, equation [11, eq. 8] takes the form

$$C_l := nose + tube = 1.02 c_f \left[1 + \frac{3}{2(L/d)^{\frac{3}{2}}}\right]\frac{A_{ws}}{A_{cs}} \qquad (50)$$

where c_f is the skin friction coefficient and L/d the length to diameter ratio of the rocket.

Base and fins of the rocket

Flow separation causes low pressure at the rear of the rocket which results in base drag. An approximate equation for the base drag is given by

$$base = \frac{0.029}{\sqrt{nose + tube}}, \qquad (51)$$

see [11, eq. 9]. Fins cause additional friction drag, pressure drag and induced drag. But they substantially enhance the flight stability of our rocket. Generally, the fin drag depends on various parameters like the thickness to chord ratio, planform area and cross-sections. We only present a rough estimation in this paper. In order to get some upper limit we use fairly pessimistic assumptions for the zero lift fin drag coefficient. Following [11, p. 17], the fin thickness to chord ratio will rarely be greater than 0.1 for a typical model rocket. Average values for the zero lift fin drag

coefficient are given in [11, fig. 40] for rectangular, rounded and steamlined cross-sections. Certainly, we cannot choose a steamlined[4] cross-section for a pessimistic estimate. Let τ denote the thickness to chord ratio and fin* the zero lift fin drag coefficient in the following. The plotted data in [11, fig. 40] suggests that fin* depends linearly on τ in the range of $0.03 < \tau < 0.136$. Within this interval we deduced the following relations for rectangular (52) and rounded (53) cross-section:

$$fin* \approx 0.875\tau, \tag{52}$$

$$fin* \approx 0.5\tau. \tag{53}$$

Figure 4 shows that the graphic illustration of (52) and (53) reproduces the corresponding lines in [11, fig. 40]. Let us assume in the following that the fins have rectangular cross-section. The zero lift fin drag coefficient fin* is based on the fin area A_{fin} whereas the other drag coefficients are based on the body tube cross-sectional area A_{cs}. Therefore, we have to adjust the coefficient fin* by

$$fin = fin* \frac{A_{fin}}{A_{cs}} = 0.875\frac{A_{fin}}{A_{cs}}\tau. \tag{54}$$

As above mentioned, rocket body and fins jointly cause additional interference drag (int). Due to [11, eq. 21], the interference drag can be approximated by

$$int = \frac{fin* C_R d}{2A_{cs}}N = 0.875\frac{C_R d}{2A_{cs}}N\tau \tag{55}$$

where C_R is the root chord of the fin, d the diameter of the body tube, A_{cs} again its cross-sectional area, and N is the number of fins. The more accurate drag calculation given in [11] also includes drag from launch lugs. The examples given in [11, p. 44, 49] lead to a comparatively small launch lugs drag coefficient in the range of about 0.02 to 0.03 . For our rough estimation we incorporate this kind of drag by adding 0.03 .
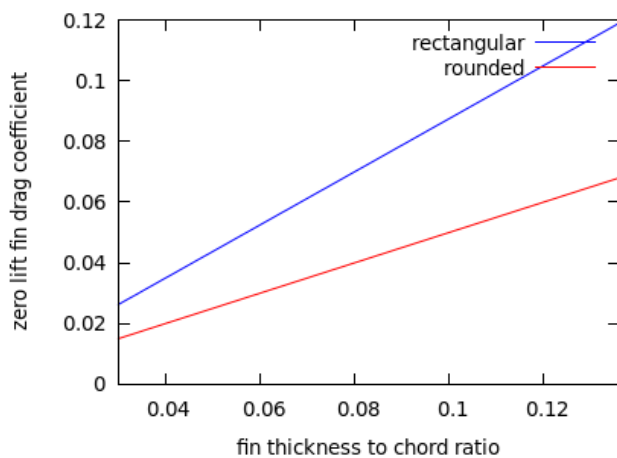
Figure 4.   Zero lift fin drag coefficients for fins with rectangular and rounded cross-sections.

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[4] According to [11, fig. 41], the zero lift fin drag coefficient for steamlined cross-sections at ratio 0.1 does not exceed roughly 0.019 (laminar) to 0.024 (turbulent) for 30 m/s (100 ft/sec).

Total drag coefficient

Collectiong the above discussion, we receive a base value for the drag coefficient of the model rocket by

$$c_D = nose + tube + base + fin + int + lau \tag{56}$$

However, we have not considered the surface texture of the rocket so far, "*roughness can cause drag to increase by about 25 percent*", c.f. [11, p. 43]. Since our rockets nose consists of half a tennis ball, fins and launch lugs are fixed with hot glue and cellotape, we presuppose that the surface might be rather rough. This leads to a significant uncertainty which we incorporate as an error in the following. Finally, we receive a more or less adequate formula for the drag coefficient of our D.I.Y. rocket by

$$c_D = \left[C_l + \frac{0.029}{\sqrt{C_l}} + \frac{0.875\tau}{A_{cs}}\left(A_{fin} + \frac{C_R d}{2}\right)N + 0.03\right] \\ \times (1.125 \pm 0.125) \tag{57}$$

with C_l from (50). Beside the ingredients that enter into C_l, τ is the thickness to chord ratio of the fins, C_R the root chord of the fin, and N the number of the fins. The skin friction coefficient c_f depends on the kind of air flow and varies greatly with the Reynolds number, see right side of Figure 5.
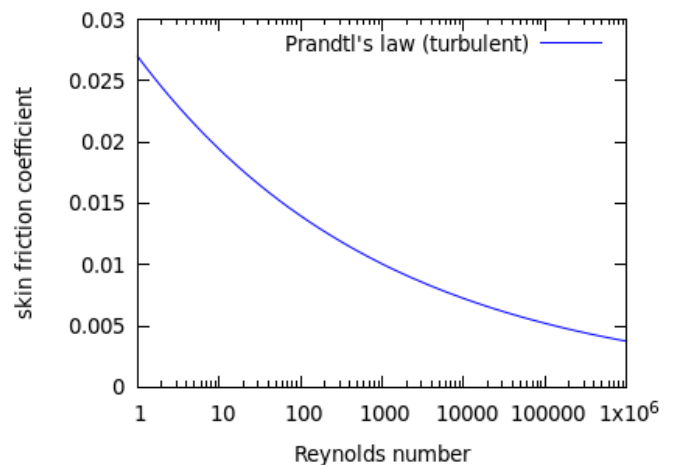
Figure 5.   Skin friction coefficient as a function of Reynolds number given by Prandtl's law, see [25, eq. 7.43].

As c_f depends on the speed of the rocket it would be the best to include the skin friction into the system of differential equations (27). However, our estimation of the total drag coefficient seems to be too vague to justify this approach. Therefore, we decided to use a constant value for the skin friction coefficient in the following estimation.

Estimation of the drag coefficient for our D.I.Y. rocket

Now let us try a rough estimation for the drag coefficient by (57). Our rockets maximum speed lies in the range of $17 ms^{-1}$. The characteristic length of our rocket lies in the range of 0.35 m. Hence, a kinematic air viscosity of about $1,5 \cdot 10^{-5} m^2 s^{-1} (at 15^o C)$ leads to a maximum Reynolds number of roughly $4 \cdot 105$. A mean value for the skin friction coefficient within this range based on Prandtl's law [25, eq. 7.43] is given by

$$\bar{c}_f = \frac{1}{4 \cdot 10^5} \int \frac{0.027}{(Re_x)^{1/7}} dRe_x \approx 0.005. \qquad (58)$$

Indeed, this value is in accordance with the skin friction coefficient at a Reynolds number of $4 \cdot 10^5$ given in [11, fig. A-1]. Our rocket has a diameter of 8 cm which leads to a cross-sectional area of $A_{cs} \approx 0.005 m^2$. The wetted surface area of the rocket (roughly a cylinder with patched-up half of a tennis ball[5]) amounts to $A_{ws} \approx 0.1 m^2$. The rocket has N=3 fins. Each has a root chord of 7.5 cm and a fin area of $A_{fin} \approx 0.0035 m^2$. The fins thickness to chord ratio is roughly 0.08. The drag coefficient calculated from (57) is

$$c_D = 0.57 \pm 0.06. \qquad (59)$$

For a wxMaxima source code see again https://github.com/tguent/code.

Wind tunnel experiment at TU Dortmund University

Prof. Dr. Andreas Brümmer of the chair of Fluidics at TU Dortmund University kindly provided us the opportunity to check our estimations for the drag coefficient in a wind tunnel. The drag force was measured with a dynamometer that has an error of about 0.05 N. We always had to do an additional measurement for the calibration. Thus, in the worst-case scenario our error adds up to $\Delta FD = 0.1N$. The wind speed was given to the first decimal place. Hence, we use an error of $\Delta v = 0.05 ms^{-1}$ in the following considerations. The drag coefficient is related to drag force and wind speed by $c_D = \frac{2 F_D}{\rho_{air} A_R v^2}$. Accordingly, the error of the drag coefficient can be estimated by

$$\Delta c_D = \left| \frac{\partial c_D}{\partial F_D} \right| \Delta F_D + \left| \frac{\partial c_D}{\partial v} \right| \Delta v$$
$$\frac{2 \Delta F_D}{\rho_{air} A_R v^2} + \frac{4 F_D \Delta v}{\rho_{air} A_R v^3} = c_D \left(\frac{\Delta F_D}{F_D} + 2 \frac{\Delta v}{v} \right) \qquad (60)$$

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[5] Drag coefficients of a nonspinning tennis balls have been measured to about 0.6 to 0.7 , see [13].

Table 1 . Drag coefficient from Wind tunnel Experiment:

Drag Force F_D with an error of $\Delta FD = 0.1N$, wind speed v with an error of $\Delta v = 0.05 ms^{-1}$.

F_D (N)	0.2	0.2	0.25
v (m/s)	9.6	11.3	12.8
c_D	0.7 ± 0.4	0.5 ± 0.3	0.5 ± 0.2
F_D (N)	0.3	0.4	0.4
v (m/s)	13.8	14.6	15.9
c_D	0.5 ± 0.2	0.6 ± 0.2	0.5 ± 0.1
F_D (N)	0.45	0.6	0.6
v (m/s)	16.8	18.3	18.9
c_D	0.5 ± 0.1	0.6 ± 0.1	0.5 ± 0.1
F_D (N)	0.65	0.7	-
v (m/s)	19.8	20	-
c_D	0.6 ± 0.1	0.6 ± 0.1	-

Altogether we receive a drag coefficient of

$$c_D = 0.6 \pm 0.2 \qquad (61)$$

which is an astonishingly good value for our D.I.Y. rocket. But as Figure 6 shows, the streamline flow pattern suggests that our rockets aerodynamic properties are not bad.
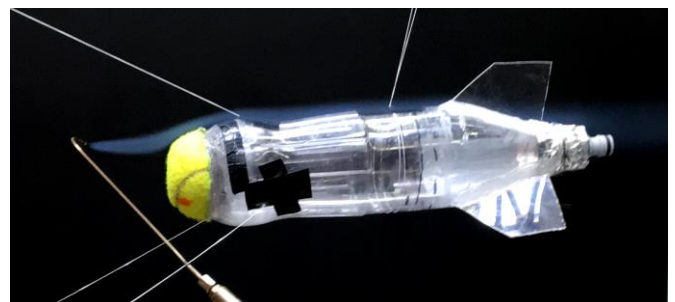

Figure 6.   Streamline flow pattern made visible by smoke in the wind tunnel at departement of Fluidics at TU Dortmund Universiy

X.   THE ROCKET LAUNCH EXPERIMENT

The rocket launch was repeated with different initial values for pressure, amount of water, and temperature (of water). The model rocket which was used for the experiment has a weight of 143 g. It was constructed from a bottle with a volume of $1 dm^3$. At launch the rocket was filled with an amount of V_l water and V_0 air at pressure p_0 and temerature

T. The difference to atmospheric pressure is $p_0 - p_a$. The Pressure p_0 was measured with an error of $\Delta p = \pm 0.05$ bar . The error is based on the scale of the manometer. The rocket has a radius of 4 cm. We drew a scala for the water level which is legible up to about ± 2 mm. Within the relevant scope, the rocket is cylindric (in a good approximation). Accordingly, the initial amount of water V_l (and therewith the air volume V_0) is determined up to an error of $\Delta V = \pm 10$ ml. Due to the precision of the temperature measurements the error of temperature does not affect the results. Determining the rockets altitude h_{mes} makes up the major part in measuring inaccuracy within our experiment. We deployed a measuring rod beside the launching device. The whole flight phase was filmed (We provide a link to the corresponding videos, see below.). By evaluating the videos we receive the maximum altitude of the rocket in comparison with the length of the measuring rod. The extrapolation leads to an error of ± 0.5 m. The messured altitude can be confirmed by the given video link.

It is the aim of this section to compare the experimental data with the theoretical results. We use the averages of the experimental data for the theoretical results. These are calculated as described in Section VIII. Thereby, h_{calc} is calculated by using the system of differential equations (27) for the thrust phase. We receive h_{est} by estimating the part of the altitude that is obtained from the water thrust phase from equation (47). This estimation can be done with a graphing calculator only. The rockets drag coefficient was set to $cD = 0.6$. For the air thrust efficiency factor we use μ=3.

Table 2. The experimental data

	$T(C)$	$V_l(ml)$	$V_0(ml)$	$p_0(bar)$	$h_{mes}(m)$
1	17	350 ± 10	650 ± 10	3.5 ± 0.05	17.5 ± 0.5
2	17	325 ± 10	675 ± 10	3.5 ± 0.05	17.1 ± 0.5
3	17	255 ± 10	745 ± 10	3.0 ± 0.05	14.5 ± 0.5
4	17	255 ± 10	745 ± 10	3.5 ± 0.05	17.4 ± 0.5
5	20	empty	1000	2.5 ± 0.05	> 2
6	35	250 ± 10	750 ± 10	2.5 ± 0.05	7.6 ± 0.5

Table 3. Video links

1	https://youtu.be/YXGb75eOqbI
2	https://youtu.be/9XY_6iSHXPE
3	https://youtu.be/3l5JJHO0Tp8
4	https://youtu.be/RUkJwDCq8uU
5	https://youtu.be/MifjHZH3Z7Y
6	https://youtu.be/8MRufGWCUjM

Table 4. Theoretical predictions for the mean values

	$T(C)$	$V_l(ml)$	$p_0(bar)$	$h_{mes}(m)$	$h_{est}(m)$
1	17	350	3.5	17.2	17.5
2	17	325	3.5	17.4	17.6
3	17	255	3.0	13.3	13.4
4	17	255	3.5	17.2	17.3
5	20	< 1	2.5	2.4	2.4
6	35	250	2.5	9.5	9.5

A wxMaxima source code for the above calculations is available at https://github.com/tguent/code.

XI. CONCLUSION

Obviously, the method introduced in this paper (most commonly) yields accurate theoretical predictions. The predictions in case 1, 2 and 4 lie within the experimental error. Launch 5 is hardly to evaluate because the rocket crashed into the ceiling. Indeed, the main goal of launch 5 was to show that there is significant thrust if the rocket is filled with pressured air only. Launch 3 yields an altitude of 13.6 m at the lower boundary which differs from the predicted value of 13.4 m . The question comes up whether the error in altitude has to be enlarged. On the one hand, the position of the camera has to be far enough from the experiment that the elevation angle remains small. On the other hand, huge distance leads to more blurred pictures. It stands to reason that one has to find a more suitable method to evaluate the maximum altitude. In case of launch no. 6, theoretical and experimental results does not coincide. Indeed, we launched the rocket several times without getting exploitable data because the rocket does not lift off correctly. In some cases the rocket stuck to long to the launching device due to friction. In other cases the rocket lurches through the air. Furthermore, our model D.I.Y. rocket begins to be leaking after a huge amount of experiments. Putting it all together, our method is suitable to predict the altitude in case that the rocket lifts off perfectly. But even in this case, the water rocket physics represent a highly chaotic system. On this background, the simple estimation proposed in this paper yields amazingly good results.

REFERENCES

[1] M. Abramowitz, I. A. Stegun, "*Handbook of mathematical functions with formulas, graphs, and mathematical tables*", Dover Publications, 1965.

[2]   R. Barrio-Perotti, E. Blanco-Marigorta, K. Arguelles-Diaz, J. Fernandez-Oro, *"Experimental Evaluation of the Drag Coefficient of Water Rockets by a Simple Free-Fall Test"*, European Journal of Physics, v30 n5 pp.**1039-1048**, **2009**.

[3]   Barrio-Perotti, E. Blanco-Marigorta, J. Fernández-Francos, M. Galdo-Vega, "*Theoretical and experimental analysis of the physics of water rockets*", European Journal of Physics Vol.**31**, pp. **1131, 2010**.

[4]   I. N. Bronstein "*Taschenbuch der Mathematik"*, B.G.Teubner Stuttgart-Leipzig, ISBN-3-8154-2001-6, 1996.

[5]   A. L. Buck, "*New equations for computing vapor pressure and enhancement factor*", J. Appl. Meteorol., Vol.**20**, pp. **1527–1532**, **1981**.

[6]   T. A. Campbell, M. Okutsu, "*Model Rocket Projectfor Aerospace Engineering Course: Trajectory Simulation and Propellant Analysis*", arXiv:1708.01970 [physics.ed-ph], **2017**.

[7]   L. J. Clancy, "*Aerodynamics"*, Wiley, ISBN 978-0-470-15837-1, 1975.

[8]   M. Clifford, K. Simmons, P. Shipway, "*An Introduction to Mechanical Engineering"*, CRC Press, **2006**.

[9]   G. A. Finney, "*Analysis of a water-propelled rocket: A problem in honors physics*", American Journal of Physics Vol.**68**, pp. **223**, **2000**.

[10]   C.J. Gommes, "*A more thorough analysis of water rockets: Moist adiabats, transient flows and inertial forces in a soda bottle."*, American Journal of Physics Vol.**78**, pp. **236**, https://doi.org/10.1119/1.3257702, **2010**.

[11]   G. M. Gregorek, "*Aerodynamic Drag of Model Rockets*.", Estes Industries, Penrose, CO, pp. **2-51**, **1970**.

[12]   International Civil Aviation Organization, "*Manual of the ICAO Standard Atmosphere*.", Doc 7488-CD, Third Edition, ISBN 92-9194-004-6, **1993**.

[13]   R. Mehta, F. Alam, A. Subic, "*Review of tennis ball aerodynamics, Sports Technology."* 1:1, pp. **7-16**, DOI: 10.1080/19346182.2008.9648446, **2008**.

[14]   D. Kagan, L. Buchholtz, L. Klein, "*Soda-bottle water rockets"*, Phys. Teach. Vol. **33**, pp. **150**, **1995**.

[15]   Messerschmid und S. Fasoulas, "*Die Ziolkowsky-Raketengleichung", In: Raumfahrtsysteme*, https://doi.org/10.1007/978-3-662-49638-1_2, Springer Vieweg, Berlin, Heidelberg, 2017.

[16]   J. Moran, "*An Introduction to Theoretical and Computational Aerodynamics*.", Wiley, New York, 1984.

[17]   R.A. Nelson, M.E. Wilson, "*Mathematical analysis of a model rocket trajectory Part I: The powered phase."* Phys. Teach. Vol. **14**, pp. **150–161**, **1976**.

[18]   J.M. Prusa, "*Hydrodynamics of a Water Rocket."*, Siam Rev., Vol. **42**, pp. **719**, **2000**.

[19]   A. Romanelli, I. Bove, F.G. Madina, "*Air expansion in the water rocket."*, American Journal of Physics Vol. **81**, pp. **762**, https://doi.org/10.1119/1.4811116, **2013**.

[20]   Simple Drag Tests for Water Rockets – seeds2lrn.com, http://fliphtml5.com/rftx/obtf, visited 05.03.2019.

[21]   L. J. Slater, "*Generalized Hyperbolic Functions*.", Cambridge University Press, 1966.

[22]   H. Stöcker, "*Taschenbuch der Physik"*, Verlag Harri Deutsch, 1998

[23]   K. E. Tsiolkovsky, "*The Exploration of Cosmic Space by Means of Reaction Devices."*, The Science Review (in Russian) (5), **1903**.

[24]   A. Vodopivec, "*wxMaxima 18.02.0.*", http://andrejv.github.io/wxmaxima/

[25]   F. M. White: "*Fluid Mechanics"*, McGraw-Hill, ISBN 978-0-07-352934-9, seventh edition, 2011.