Fuzzy q-b Open Sets and Fuzzy q-b Separation in Fuzzy Quad Topological Space

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Available online at: www.isroset.org

Received: 28/May/2018, Revised: 07/June/2018, Accepted: 21/June/2018, Online: 30/June/2018

Abstract - The main aim of this paper is to study fuzzy q-b-open sets in fuzzy quad topological spaces along with their several properties and characterization. We study fuzzy q-open sets, fuzzy q-gb open sets, fuzzy q-b continuous function and fuzzy q-b separation and obtain some of their basic properties.

Keywords: Fuzzy quad topology, fuzzy q-b open sets, fuzzy q-gb open sets, fuzzy q-b continuous function, fuzzy q-b separation.

AMS Mathematics Subject Classification (2010): 54A40

I. INTRODUCTION


Abu-Safiya A.S. and fora A.A. [15] studied Fuzzy Separation axioms and fuzzy continuity in fuzzy bitopological spaces. Das N. R. and Baishya P.C. [16] studied Fuzzy bitopological spaces and separation axioms. Palaniammal S. [5] introduced fuzzy tri topological space. We [17] introduced fuzzy tri-b open sets and fuzzy tri-b separation in fuzzy tri topological space. The purpose of the present paper is to introduced fuzzy q-b-open sets in fuzzy quad topological space and studied their fundamental properties in fuzzy quad topological space. We study fuzzy q-b continuous, fuzzy q-b separation and obtain some of their basic properties.

II. PRELIMINARIES

Definition 2.1[1]: Let A be a fuzzy set of a fuzzy topological spaces X. Then A is called a fuzzy b-open set if \( A \subseteq \text{pcl}(p\text{int}A) \).

Definition 2.1[13]: Let \((X, T_1, T_2, T_3)\) be a tri topological space, a subset A of a space X is said to be tri-b open set if \( S \subseteq \text{tri} - \text{cl}(\text{tri} - \text{int}S) \cup \text{tri} - \text{int}(\text{tri} - \text{cl}S) \).
Definition 2.3[16]: Let $X$ be a non-empty set. Let $F_1, F_2, F_3$ be fuzzy topologies on $X$. Then $(X, T_1, T_2, T_3)$ is called a fuzzy tri topological spaces.

III. FUZZY Q-B OPEN SETS IN FUZZY QUAD TOPOLOGICAL SPACE

Theorem 3.1: Let $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ be a fuzzy quad topological space, and $\mathcal{X}_\lambda \leq \tilde{I}_X$. Then

(a) $\tilde{I}_X - (F_q - bcl(\mathcal{X}_\lambda)) = (F_q - bcl(\tilde{I}_X - \mathcal{X}_\lambda))$.
(b) $\tilde{I}_X - (F_q - bint(\mathcal{X}_\lambda)) = (F_q - bcl(\tilde{I}_X - \mathcal{X}_\lambda))$.

Proof: Let $\mathcal{X}_\lambda \leq \tilde{I}_X$, where $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ is a fuzzy quad topological space.

(a) Now, $F_q - bcl(\mathcal{X}_\lambda) = \land \{\mathcal{X}_\delta : \mathcal{X}_\lambda \prec \mathcal{X}_\delta \text{ and } \mathcal{X}_\delta \text{ is fuzzy q-b closed set}\}^c$

$= \lor \{\mathcal{X}_\delta^c : \mathcal{X}_\delta^c \prec \mathcal{X}_\lambda^c \text{ and } \mathcal{X}_\delta^c \text{ is fuzzy q-open set}\}$

$= F_q - bint(\mathcal{X}_\lambda^c)$

(b) Similarly, $(F_q - bint(\mathcal{X}_\lambda))^c = (F_q - bcl(\mathcal{X}_\lambda^c))$.

Proposition 3.2: Let $\mathcal{X}_\lambda$ be a subset of a space $X$. Then

1. $Fpscl(\mathcal{X}_\lambda) = \mathcal{X}_\lambda \lor F_q - cl(F_q - int(t(F_q - cl(\mathcal{X}_\lambda))))$.
2. $Fpsin(t(\mathcal{X}_\lambda)) = \mathcal{X}_\lambda \land F_q - int(F_q - cl(F_q - int(t(\mathcal{X}_\lambda))))$
3. $Fpscl(p \sin(t(\mathcal{X}_\lambda))) = F_q - cl(F_q - int(t(\mathcal{X}_\lambda))) \land
4. Fpsin(t(pscl(\mathcal{X}_\lambda))) = F_q - int(F_q - cl(\mathcal{X}_\lambda))$

Theorem 3.3: In a fuzzy quad topological space $(X, \tau_1, \tau_2, \tau_3, \tau_4)$, for a subset $\mathcal{X}_\lambda$ of $X$, the following are equivalent:

1. $\mathcal{X}_\lambda$ is fuzzy q-b open.
2. $\mathcal{X}_\lambda = Fp \int \mathcal{X}_\lambda \lor F \sin t \mathcal{X}_\lambda$
3. $\mathcal{X}_\lambda \leq Fp \ cl (Fp \ int \mathcal{X}_\lambda)$

Proof: Let $\mathcal{X}_\lambda \leq \tilde{I}_X$, where $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ a fuzzy quad topological space.
(1) ⇒ (2) suppose that $\mathcal{X}_\lambda$ is a fuzzy q-b open set.

So $\mathcal{X}_\lambda < (Fq - cl(Fq - int \mathcal{X}_\lambda)) \vee (Fq - int(Fq - cl \mathcal{X}_\lambda))$. Now,

$$Fp int \mathcal{X}_\lambda \vee F \sin t \mathcal{X}_\lambda = \{\mathcal{X}_\lambda \wedge Fq - int(Fq - cl \mathcal{X}_\lambda)\} \vee \{\mathcal{X}_\lambda \wedge Fq - cl(Fq - int \mathcal{X}_\lambda)\}$$

[By proposition (3.2) (2) and (3)]

$$= \mathcal{X}_\lambda \wedge \{Fq - int(Fq - cl \mathcal{X}_\lambda)\} \vee \{Fq - cl(Fq - int \mathcal{X}_\lambda)\}$$

$$= \mathcal{X}_\lambda$$

∴ $\mathcal{X}_\lambda = Fp int \mathcal{X}_\lambda \vee F \sin t \mathcal{X}_\lambda$.

(2) ⇒ (3) Let $\mathcal{X}_\lambda = Fp int \mathcal{X}_\lambda \vee Fs int \mathcal{X}_\lambda$ [by Proposition (3.2) (2) and (3), we have $\mathcal{X}_\lambda = Fp int \mathcal{X}_\lambda \vee (\mathcal{X}_\lambda \wedge Fq - cl(Fq - in t \mathcal{X}_\lambda))$]

$$\leq Fp cl(Fp int \mathcal{X}_\lambda), \text{i.e. } \mathcal{X}_\lambda \leq Fp cl(Fp int \mathcal{X}_\lambda).$$

(3) ⇒ (1) Let $\mathcal{X}_\lambda \leq Fp cl(Fp int \mathcal{X}_\lambda)$. Then $\mathcal{X}_\lambda \leq Fp int \mathcal{X}_\lambda \vee Fq - cl(Fq - in t \mathcal{X}_\lambda)$ [by Proposition (3.2) (2)]

i.e. $\mathcal{X}_\lambda \leq \mathcal{X}_\lambda \wedge Fq - int(Fq - cl \mathcal{X}_\lambda) \vee (\mathcal{X}_\lambda \wedge Fq - cl(Fq - int \mathcal{X}_\lambda))$

[By Proposition (3.2) (3)]

$$= \{\mathcal{X}_\lambda \vee Fq - cl(Fq - int \mathcal{X}_\lambda)\} \wedge \{Fq - int(Fq - cl \mathcal{X}_\lambda) \vee Fq - cl(Fq - int \mathcal{X}_\lambda)\}$$

i.e. $\mathcal{X}_\lambda$ is a fuzzy q-b open set.

**Note 3.4:** Every fuzzy q-b open set can be represented as a union of fuzzy q-pre-open set and a fuzzy q-semi open set.

**Theorem 3.5:** If $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ be a fuzzy quad topological space, then

(a) The intersection of a fuzzy q-$\alpha$ open set and a fuzzy q-b open set is a fuzzy q-b open set.

(b) Fuzzy-$\alpha_q$ and fuzzy quad topological spaces have the same class of fuzzy q-b open set.

**Proof:** Let $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ be a fuzzy quad topological space.

(a) Let $\mathcal{X}_{\lambda}$ be a fuzzy $\alpha_q$ open set and $\mathcal{X}_{\delta}$ be a fuzzy q-b open set.

Now, $\mathcal{X}_{\mu} = \mathcal{X}_{\lambda} \wedge \mathcal{X}_{\delta}$

$$= Fq - \alpha \mathcal{X}_{\lambda} \wedge Fq - b \text{int } \mathcal{X}_{\delta}$$

$$\leq Fq - b \mathcal{X}_{\lambda} \wedge Fq - b \text{int } \mathcal{X}_{\delta}$$

$$= Fq - b \text{int}(\mathcal{X}_{\lambda} \wedge \mathcal{X}_{\delta})$$

$$= Fq - b \text{int}(\mathcal{X}_{\mu})$$

i.e. $\mathcal{X}_{\mu} \leq Fq - b \text{int}(\mathcal{X}_{\mu})$

But $Fq - b \text{int}(\mathcal{X}_{\mu}) \leq \mathcal{X}_{\mu}$
Hence $\chi_\mu = Fq - b\text{int}(\chi_\mu)$ i.e. $\chi_\mu = \chi_\lambda \land \chi_\delta$ is a fuzzy q-b open set.

**Theorem 3.6:** Let $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ be a fuzzy quad topological space and $\chi_\mu$ be a subset of $X$, then

(a) $Fq - b\text{cl}\chi_\mu = Fscl\chi_\mu \land Fpcl\chi_\mu$.

(b) $Fq - b\text{int}\chi_\mu = F\text{int}\chi_\mu \lor Fp\text{int}\chi_\mu$.

**Proof:** Let $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ be a fuzzy quad topological space and $\chi_\mu \leq 1_X$.

(a) Since $Fq - b\text{cl}\chi_\mu$ is a fuzzy q-b closed set.

Hence $Fq - \text{int}(Fq - cl(Fq - Fbcl\chi_\mu)) \land Fq - cl(Fq - \text{int}(Fq - Fbcl\chi_\mu)) \leq Fq - Fbcl\chi_\mu$ Again,$\newline\newline Fq - \text{int}(Fq - cl(Fq - Fbcl\chi_\mu)) \land Fq - cl(Fq - \text{int}\chi_\mu) \leq \newline \newline Fq - \text{int}(Fq - cl(Fq - Fbcl\chi_\mu)) \land Fq - cl(Fq - \text{int}(Fq - Fbcl\chi_\mu))$

i.e. $Fq - \text{int}(Fq - cl\chi_\mu) \land Fq - cl(Fq - \text{int}\chi_\mu) \leq Fq - bcl\chi_\mu$.

i.e. $\chi_\mu \lor Fq - \text{int}(Fq - cl\chi_\mu) \land Fq - cl(Fq - \text{int}\chi_\mu) \leq \chi_\mu \lor Fq - bcl\chi_\mu$.

i.e., $Fscl\chi_\mu \land Fpcl\chi_\mu \leq Fq - bcl\chi_\mu$ ------------------------ (i)

Next, i.e., $Fq - bcl\chi_\mu \leq Fscl\chi_\mu$ and $Fq - bcl\chi_\mu \leq Fpcl\chi_\mu$

i.e., $Fq - bcl\chi_\mu \leq scl\chi_\mu \land pcl\chi_\mu$ ------------------------ (ii)

From (i) & (ii), it follows that

$Fq - bcl\chi_\mu = scl\chi_\mu \land pcl\chi_\mu$

(b)Since $Fq - b\text{int}\chi_\mu$ is a $Fq - b$ open set, we have

$Fq - cl(Fq - \text{int}(Fq - b\text{int}\chi_\mu)) \lor Fq - \text{int}(Fq - cl(Fq - b\text{int}\chi_\mu)) \geq Fq - b\text{int}\chi_\mu$

Again,$\newline\newline Fq - cl(Fq - \text{int}(Fq - b\text{int}\chi_\mu)) \lor Fq - \text{int}(Fq - cl(Fq - b\text{int}\chi_\mu)) \newline \newline \leq Fq - cl(Fq - \text{int}\chi_\mu)$

i.e. $Fq - b\text{int}\chi_\mu \leq (Fq - cl(Fq - \text{int}\chi_\mu)) \lor Fq - \text{int}(Fq - cl\chi_\mu)$

i.e. $\chi_\mu \land Fq - b\text{int}\chi_\mu \leq (\chi_\mu \land Fq - cl(Fq - \text{int}\chi_\mu)) \lor (\chi_\mu \land Fq - \text{int}(Fq - cl\chi_\mu))$

i.e. $Fq - b\text{int}\chi_\mu \leq s\text{int}\chi_\mu \lor p\text{int}\chi_\mu$ ------------------------ (i)

Next, $Fs\text{int}\chi_\mu \leq Fq - b\text{int}\chi_\mu$ and $Fp\text{int}\chi_\mu \leq Fq - b\text{int}\chi_\mu$.

i.e. $Fs\text{int}\chi_\mu \lor Fp\text{int}\chi_\mu \leq Fq - b\text{int}\chi_\mu$ ------------------------ (ii)

From (i) and (ii), it follows that

$Fq - b\text{int}\chi_\mu = Fs\text{int}\chi_\mu \lor Fp\text{int}\chi_\mu$. 
Theorem 3.7: If \( \mathcal{X}_\lambda \) be a fuzzy subset of a fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\), then
\[
Fq - \text{bint} (Fq - \text{bcl} \mathcal{X}_\lambda) = Fq - \text{bcl} (Fq - \text{int} \mathcal{X}_\lambda).
\]

Proof: Let \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) be a fuzzy quad topological space. Now, \(Fq - \text{bint} (Fq - \text{bcl} \mathcal{X}_\lambda) = Fsint(Fq - \text{bcl} \mathcal{X}_\lambda) \lor Fp \text{int} (Fq - \text{bcl} \mathcal{X}_\lambda).
\]

\& \(Fq - \text{bcl} (Fq - \text{bint} \mathcal{X}_\lambda) = Fq - \text{bcl} (Fq - \text{bcl} (Fsint \mathcal{X}_\mu) \lor Fp \text{int} \mathcal{X}_\mu)
\]
\[
= Fq - \text{bcl} (Fq - \text{bcl} (Fsint \mathcal{X}_\mu) \lor Fq - \text{bcl} (Fsint \mathcal{X}_\mu))
\]
\[
= Fq - \text{bcl} (Fsint \mathcal{X}_\mu) \lor Fp \text{int} (Fp \text{cl} \mathcal{X}_\mu).
\]

Hence from (i) and (ii),
\[
Fq - \text{bint} (Fq - \text{bcl} \mathcal{X}_\lambda) = Fq - \text{bcl} (Fq - \text{bcl} \mathcal{X}_\lambda).
\]

Theorem 3.8: A fuzzy subset \( \mathcal{X}_\lambda \) in a fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) is fuzzy q-b open if and only if there exists a fuzzy quad pre-open set \( \mathcal{X}_\delta \) in \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) such that \( \mathcal{X}_\delta \leq \mathcal{X}_\lambda \leq \text{Fpcl} (\mathcal{X}_\delta) \).

Proof: Let \( \mathcal{X}_\lambda \prec \tilde{1}_X \) be a fuzzy nonempty fuzzy subset of \( X \).
\[
\mathcal{X}_\lambda \leq \text{Fpcl} (Fp \text{int} \mathcal{X}_\lambda) \hspace{1cm} (1)
\]

Now, as usual \( Fp \text{int} \mathcal{X}_\lambda \leq \mathcal{X}_\lambda \) and \( \mathcal{X}_\delta = Fp \text{int} (\mathcal{X}_\lambda) \) is a fuzzy quad pre-open set. Hence, from (1) it follows that \( \mathcal{X}_\delta \leq \mathcal{X}_\lambda \leq \text{Fpcl} (\mathcal{X}_\delta) \).

Conversely, for a fuzzy set \( \mathcal{X}_\lambda \) there exists fuzzy quad pre-open set \( \mathcal{X}_\delta \) such that \( \mathcal{X}_\delta \leq \mathcal{X}_\lambda \leq \text{Fpcl} (\mathcal{X}_\delta) \).

Since, \( Fp \text{int} (\mathcal{X}_\lambda) \) is the maximal fuzzy quad pre-open set contained in \( \mathcal{X}_\lambda \).

Hence \( \mathcal{X}_\delta \leq Fp \text{int} (\mathcal{X}_\lambda) \leq \mathcal{X}_\lambda \hspace{1cm} (3) \)

Now, \( pcl (\mathcal{X}_\delta) \leq \text{Fpcl} (Fp \text{int} \mathcal{X}_\lambda) \hspace{1cm} (4) \) [from (3)]

Combining (2) and (3) we get
\[
\mathcal{X}_\lambda \leq \text{Fpcl} (Fp \text{int} \mathcal{X}_\lambda),
\]
which means that \( \mathcal{X}_\lambda \) is a fuzzy q-b open set.

Corollary 3.9: A subset \( \mathcal{X}_\lambda \) in a fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) is fuzzy q-b open if and only if it contains fuzzy quad pre-open set but not its fuzzy quad pre-closure.

Theorem 3.10: If \( \mathcal{X}_\lambda \) is a fuzzy q-b open set in a space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) then
\[
\mathcal{X}_\lambda \leq Fq - \text{cl} (Fq - \text{int} \mathcal{X}_\lambda) \lor Fq - \text{int} (Fq - \text{cl} \mathcal{X}_\lambda)
\]
is a fuzzy quad pre-open set.
IV. FUZZY q-b CONTINUOUS FUNCTION IN FUZZY QUAD TOPOLOGICAL SPACE

Definition 4.1: A fuzzy function \( f : (X, \tau_1, \tau_2, \tau_3, \tau_4) \rightarrow (Y, \tau'_1, \tau'_2, \tau'_3, \tau'_4) \) is said to be fuzzy q-b closed (resp. fuzzy q-b open) if for every fuzzy q-b closed (resp. fuzzy q-b open) subset \( \mathcal{X}_\lambda \) of \( X \), \( f(\mathcal{X}_\lambda) \) is fuzzy q-b closed (resp. fuzzy q-b open) in \( Y \).

Definition 4.2: Let \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) and \((Y, \tau'_1, \tau'_2, \tau'_3, \tau'_4)\) be two fuzzy quad-topological spaces. A fuzzy function \( f : \mathbb{I}^X \rightarrow \mathbb{I}^Y \) is called fuzzy quad-b open map if \( f(\mathcal{X}_\lambda) \) is fuzzy q-b open in \( Y \) for every fuzzy q-b open set \( \mathcal{X}_\lambda \) in \( X \).

Definition 4.3: Let \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) and \((Y, \tau'_1, \tau'_2, \tau'_3, \tau'_4)\) be two fuzzy quad-topological spaces. Let \( f : \mathbb{I}^X \rightarrow \mathbb{I}^Y \) be a fuzzy mapping. \( f \) is called fuzzy q-b closed map if \( f(\mathcal{X}_\lambda) \) is fuzzy q-b closed in \( Y \) for every fuzzy q-b closed set \( \mathcal{X}_\lambda \) in \( X \).

V. FUZZY q-gb OPEN SETS IN QUAD TOPOLOGICAL SPACE

Definition 5.1: A fuzzy subset \( \mathcal{X}_\lambda \) of a fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) is said to be fuzzy q-gb closed if \( Fq-bcl(\mathcal{X}_\lambda) \subset \mathcal{X}_\delta \) whenever \( \mathcal{X}_\lambda \subset \mathcal{X}_\delta \) and \( \mathcal{X}_\delta \) is fuzzy quad open set.

Remarks 5.2:
(i) The complement of fuzzy q-gb closed is fuzzy q-gb open.
(ii) The intersection of all fuzzy quad closed sets of \( \mathcal{X}_\lambda \) containing a subset \( \mathcal{X}_\delta \) of \( \mathcal{X}_\lambda \) is called fuzzy q-gb closure of \( \mathcal{X}_\delta \) and is denoted by \( Fq-gb-cl(\mathcal{X}_\delta) \). The union of all fuzzy q-gb open sets contained in \( \mathcal{X}_\delta \) denoted by \( Fq-gb-int(\mathcal{X}_\delta) \) is called fuzzy q-gb interior of \( \mathcal{X}_\delta \).

Theorem 5.3: Every fuzzy quad closed subset \( \mathcal{X}_\lambda \) of a fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) is fuzzy q-b closed.

Proof: Let \( \mathcal{X}_\lambda \subset \mathcal{X}_\delta \) is a fuzzy quad closed set, since \( Ftri int(\mathcal{X}_\lambda) \subset Fq-cl(int(\mathcal{X}_\lambda)) \), hence \( Fq-int(int(\mathcal{X}_\lambda)) \subset Fq-cl(int(int(\mathcal{X}_\lambda))) \). hence \( Fq-int(\mathcal{X}_\lambda) \subset Fqcl(int(int(\mathcal{X}_\lambda))) \) and \( Fq-int(\mathcal{X}_\lambda) \subset Fq-int(Fq-cl(Fq-int(\mathcal{X}_\lambda))) \vee Fq-cl(Fq-int(Fq-int(\mathcal{X}_\lambda))) \) hence \( Fq-int(\mathcal{X}_\lambda) \) is fuzzy quad-b open set, hence \( \mathcal{X}_\lambda \) is fuzzy q-b open set.

Theorem 5.4: Every fuzzy quad-b closed subset \( \mathcal{X}_\lambda \) of a fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) is a fuzzy quad-gb closed.
VI. FUZZY q- SEPARATION AXIOMS IN FUZZY QUAD TOPOLOGICAL SPACE

Definition 6.1: A fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) is said to be \(Fq - T_0\) space if and only if to give any pair of distinct points \(x, y\) in \(X\), there exists a fuzzy q-open set containing one of the points but not the other.

Example 6.2: Let \(X = \{1, 2, 3, 4\}\) be a non-empty fuzzy set, consider four fuzzy topologies on \(X\)
\[
\tau_1 = \{\tilde{1}_X, \tilde{0}_X, \chi_{\{1\}}\}, \quad \tau_2 = \{1_X, \tilde{0}_X, \chi_{\{1\}}, \chi_{\{1,3\}}\},
\tau_3 = \{\tilde{1}_X, \tilde{0}_X, \chi_{\{1\}}, \chi_{\{1,2\}}\},
\tau_4 = \{1_X, \tilde{0}_X, \chi_{\{4\}}, \chi_{\{1,4\}}\}
\]
Fuzzy q-open sets of fuzzy quad topological spaces are union of all fuzzy quad topologies.

Then fuzzy quad open sets of \(X = \{\tilde{1}_X, \tilde{0}_X, \chi_{\{1\}}, \chi_{\{4\}}, \chi_{\{1,3\}}, \chi_{\{1,2\}}, \chi_{\{1,4\}}\}\)
So \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) is \(Fq - T_0\) space.

Theorem 6.3: If \(\chi_{\{x_{i}\}}\) is a fuzzy q-open for some \(\chi_{\{x_{i}\}} \leq \chi_{\lambda}\) then \(\chi_{\{x_{i}\}} \leq Fq - cl(\chi_{\{x_{i}\}})\), for all \(\chi_{\{y_{i}\}} \neq \chi_{\{x_{i}\}}\).

Proof: Let \(\chi_{\{x_{i}\}}\) be a fuzzy q-open for some \(\chi_{\{x_{i}\}} \leq \chi_{\lambda}\), then \(\chi_{\lambda} - \chi_{\{x_{i}\}}\) is fuzzy q-closed. If \(\chi_{\{x_{i}\}} \leq Fq - (\chi_{\{y_{i}\}})\), for some \(\chi_{\{y_{i}\}} \neq \chi_{\{x_{i}\}}\), then \(\chi_{\{y_{i}\}}, \chi_{\{x_{i}\}}\) both are in all the fuzzy q-closed sets containing \(\chi_{\{y_{i}\}}\), so \(\chi_{\{x_{i}\}} \leq \chi_{\lambda} - \chi_{\{x_{i}\}}\) which is contradiction, hence \(\chi_{\{x_{i}\}} \leq Fq - (\chi_{\{y_{i}\}})\).

Theorem 6.4: In any fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\), any fuzzy distinct points have distinct fuzzy q-closure.

Proof: Let \(\chi_{\{y_{i}\}}, \chi_{\{x_{i}\}} \leq \tilde{1}_X\) with \(\chi_{\{y_{i}\}} \neq \chi_{\{x_{i}\}}\), and let \(\chi_{\lambda} = \chi_{\{x_{i}\}}\) hence \(Fq - cl(\chi_{\lambda}) = \chi_{\lambda}\) or \(\chi_{\delta}\)
Now if then \(Fq - cl(\chi_{\lambda}) = \chi_{\lambda}\) then \(\chi_{\lambda}\) is fuzzy q-closed so \(\tilde{1}_X - \chi_{\delta} = \chi_{\{x_{i}\}}\) is fuzzy q-open & not containing \(\chi_{\{y_{i}\}}\). So by theorem (6.3) \(\chi_{\{x_{i}\}} \triangleright Fq - cl(\chi_{\{y_{i}\}})\) and \(\chi_{\{y_{i}\}} \leq Fq - cl(\chi_{\{y_{i}\}})\) which implies that \(Fq - cl(\chi_{\{x_{i}\}})\) and \(Fq - cl(\chi_{\{x_{i}\}})\) are distinct. If \(Fq - cl(\chi_{\lambda}) = \tilde{1}_X\) then \(\chi_{\delta}\) is fuzzy q-open, hence \(\chi_{\{x_{i}\}}\) is fuzzy q-closed, which mean that \(Fq - cl(\chi_{\{x_{i}\}}) = \chi_{\{x_{i}\}}\) which is not equal to \(Fq - cl(\chi_{\{y_{i}\}})\).

Theorem 6.5: In any fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\), if distinct fuzzy points have distinct fuzzy q-b closure then \(\chi_{\lambda}\) is \(Fq - T_0\) space.
Proof: Let $\mathcal{X}_{[x_1]}, \mathcal{X}_{[y_1]} \leq \mathcal{X}_\lambda$ with $\mathcal{X}_{[x_1]} = \mathcal{X}_{[y_1]}$, with $Fq - b(\mathcal{X}_{[y_1]})$ is not equal to $Fq - b - cl(\mathcal{X}_{[x_1]})$, hence there exists $\mathcal{X}_{[z]} \leq \mathcal{X}_\lambda$ such that $\mathcal{X}_{[z]} \leq Fq - cl(\mathcal{X}_{[x_1]})$ but $\mathcal{X}_{[z]} \succ Fq - cl(\mathcal{X}_{[y_1]})$ or $\mathcal{X}_{[z]} \leq Fq - cl(\mathcal{X}_{[x_1]})$ but now without loss of generality, let $\mathcal{X}_{[z]} \leq Fq - cl(\mathcal{X}_{[x_1]})$.

But $\mathcal{X}_{[z]} \succ Fq - cl(\mathcal{X}_{[x_1]})$. If $\mathcal{X}_{[x]} \leq Fq - cl(\mathcal{X}_{[y_1]})$ then $Fq - cl(\mathcal{X}_{[y_1]})$ is contained in $Fq - cl(\mathcal{X}_{[y_1]})$, hence $\mathcal{X}_{[z]} \leq Fq - cl(\mathcal{X}_{[y_1]})$. which is a contradiction, this mean that $\mathcal{X}_{[x]} \succ Fq - cl(\mathcal{X}_{[y_1]})$

hence $\mathcal{X}_{[z]} \succ Fq - cl(\mathcal{X}_{[y_1]}) \mathcal{X}_{[x]} \leq Fq - cl(\mathcal{X}_{[y_1]}^c)$, hence $X$ is fuzzy $q-T_0$ space.

Definition 6.6: A fuzzy quad topological space $X$ is said to be fuzzy $q-T_1$ space if and only if for any given pair of distinct fuzzy points $\mathcal{X}_{[x_1]}, \mathcal{X}_{[y_1]}$ of $X$ there exist two fuzzy $q$-open sets $\mathcal{X}_\lambda, \mathcal{X}_\delta$ such that $\mathcal{X}_{[x_1]} \leq \mathcal{X}_\lambda, \mathcal{X}_{[y_1]} \succ \mathcal{X}_\lambda$ and $\mathcal{X}_{[y_1]} \leq \mathcal{X}_\delta, \mathcal{X}_{[x_1]} \succ \mathcal{X}_\delta$.

Theorem 6.7: Every fuzzy $q-T_1$ space is a fuzzy $q-T_0$ space.

Definition 6.8: A fuzzy quad topological space $X$ is said to be fuzzy $q-T_2$ space if and only if for $\mathcal{X}_{[x_1]}, \mathcal{X}_{[y_1]} \leq \tilde{1}_X, \mathcal{X}_{[x_1]} \neq \mathcal{X}_{[y_1]}$ there exist two disjoint fuzzy $q$-open sets $\mathcal{X}_\lambda, \mathcal{X}_\delta$ in $X$ such that $\mathcal{X}_{[x_1]} \leq \mathcal{X}_\lambda, \mathcal{X}_{[y_1]} \leq \mathcal{X}_\delta$.

Theorem 6.9: Every fuzzy $q-T_2$ space is fuzzy $q-T_1$ space.

Proof: Let $X$ is a fuzzy $q-T_2$ space and let $\mathcal{X}_{[x_1]}, \mathcal{X}_{[y_1]}$ in $X$ with $\mathcal{X}_{[x_1]} \neq \mathcal{X}_{[y_1]}$, so by hypothesis there exist two disjoint fuzzy $q$-open, say $\mathcal{X}_\lambda, \mathcal{X}_\delta$ such that $\mathcal{X}_{[x_1]} \leq \mathcal{X}_\lambda, \mathcal{X}_{[y_1]} \leq \mathcal{X}_\delta$ but $\mathcal{X}_\lambda \wedge \mathcal{X}_\delta = \tilde{0}_X$, hence $\mathcal{X}_{[x_1]} \succ \mathcal{X}_\delta, \mathcal{X}_{[y_1]} \succ \mathcal{X}_\delta$ i.e. $X$ is a fuzzy $q-T_1$ space.

Theorem 6.10: Every fuzzy $q-T_3$ space is a fuzzy $q-T_2$ space.

Proof: Let $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ be a fuzzy $q-T_3$ space and let $\mathcal{X}_{[x_1]}, \mathcal{X}_{[y_1]}$ be two distinct fuzzy points of $X$. Now by definition, $X$ is also a fuzzy $q-T_1$ space & so $\mathcal{X}_{[x_1]}$ is a fuzzy $q$- closed set. Also $\mathcal{X}_{[y_1]} \succ \mathcal{X}_{[x_1]}$. Since $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ is a fuzzy $q$-b regular space, there exist fuzzy $q$-b open sets $\mathcal{X}_\lambda$ and $\mathcal{X}_\delta$ such that $\mathcal{X}_{[x_1]} \prec \mathcal{X}_\lambda, \mathcal{X}_{[y_1]} \prec \mathcal{X}_\delta$ and $\mathcal{X}_\lambda \wedge \mathcal{X}_\delta = \tilde{0}_X$. Also $\mathcal{X}_{[x_1]} \prec \mathcal{X}_\lambda \Rightarrow \mathcal{X}_{[x_1]} \leq \mathcal{X}_\lambda$. Thus $\mathcal{X}_{[x_1]}, \mathcal{X}_{[y_1]}$ belong respectively to disjoint fuzzy $q$-b open sets $\mathcal{X}_\lambda$ and $\mathcal{X}_\delta$. According $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ is a $Fq-T_2$ space.
VII. FUZZY q-b SEPARATION AXIOMS IN FUZZY QUAD TOPOLOGICAL SPACE

Definition 7.1: A fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\) is said to be \(Fq-b-T_0\) space if and only if for a given any pair of distinct points \(\mathcal{X}_{\lambda_1}, \mathcal{X}_{\lambda_2} \in \mathcal{X}_\lambda\), there exists a fuzzy q-b open set containing one of the points but not the other.

Example 7.2: Let \(X = \{1, 2, 3, 4\}\) be a non-empty fuzzy set, consider four fuzzy topologies on \(X\) \(
\tau_1 = \{\tilde{1}_X, \tilde{0}_X, \mathcal{X}_{\{1\}}\}, \tau_2 = \{\tilde{1}_X, \tilde{0}_X, \mathcal{X}_{\{1,3\}}\}, \tau_3 = \{\tilde{1}_X, \tilde{0}_X, \mathcal{X}_{\{1,2\}}\}, \tau_4 = \{\tilde{1}_X, \tilde{0}_X, \mathcal{X}_{\{1,4\}}\}\). Then fuzzy quad open sets of fuzzy quad topological spaces are union of all fuzzy quad topologies.

Fuzzy quad-b open set of \(X\) is denoted by \(Fq-BO(X) = \{\tilde{1}_X, \tilde{0}_X, \mathcal{X}_{\{1\}}, \mathcal{X}_{\{4\}}, \mathcal{X}_{\{1,2\}}, \mathcal{X}_{\{1,3\}}, \mathcal{X}_{\{1,4\}}, \mathcal{X}_{\{2,3\}}, \mathcal{X}_{\{1,2,3\}}\}\).

Theorem 7.3: If \(\mathcal{X}_{\{y_1\}}\) is a fuzzy q-b-open for some \(\mathcal{X}_{\{y_1\}} \leq \mathcal{X}_\lambda\) then \(\mathcal{X}_{\{y_1\}} \leq Fq-b-cl(\mathcal{X}_{\{y_1\}})\), for all \(\mathcal{X}_{\{y_1\}} \neq \mathcal{X}_{\{y_1\}}\).

Proof: Let \(\mathcal{X}_{\{y_1\}}\) be a fuzzy q-b-open for some \(\mathcal{X}_{\{y_1\}} \leq \mathcal{X}_\lambda\) then \(\mathcal{X}_\lambda - \mathcal{X}_{\{y_1\}}\) is fuzzy q-b closed. If \(\mathcal{X}_{\{y_1\}} \leq Fq-b(\mathcal{X}_{\{y_1\}})\), for some \(\mathcal{X}_{\{y_1\}} \neq \mathcal{X}_{\{y_1\}}\), then \(\mathcal{X}_{\{y_1\}}\) both are in all the fuzzy q-b-closed sets containing \(\mathcal{X}_{\{y_1\}}\), so \(\mathcal{X}_{\{y_1\}} \leq \mathcal{X}_\lambda - \mathcal{X}_{\{y_1\}}\) which is contradiction, hence \(\mathcal{X}_{\{y_1\}} \leq Fq-b(\mathcal{X}_{\{y_1\}})\).

Theorem 7.4: In any fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\), any fuzzy distinct points have distinct fuzzy q-b-closure.

Proof: Let \(\mathcal{X}_{\{y_1\}}, \mathcal{X}_{\{y_1\}} \leq \tilde{1}_X\) with \(\mathcal{X}_{\{y_1\}} \neq \mathcal{X}_{\{y_1\}}\), and let \(\mathcal{X}_\lambda = \mathcal{X}_{\{y_1\}}\) hence \(Fq-b-cl(\mathcal{X}_\lambda) = \mathcal{X}_\lambda\) or \(\mathcal{X}_\delta\). Now if then \(Fq-b-cl(\mathcal{X}_\lambda) = \mathcal{X}_\lambda\) then \(\mathcal{X}_\lambda\) is fuzzy q-b-closed so \(\tilde{1}_X - \mathcal{X}_\delta = \mathcal{X}_{\{y_1\}}\) is fuzzy q-b open & not containing \(\mathcal{X}_{\{y_1\}}\).

So by theorem (7.3) \(\mathcal{X}_{\{y_1\}} > Fq-b-cl(\mathcal{X}_{\{y_1\}})\) & \(\mathcal{X}_{\{y_1\}} \leq Fq-b-cl(\mathcal{X}_{\{y_1\}})\) which implies that \(Fq-b-cl(\mathcal{X}_{\{y_1\}})\) and \(Fq-b-cl(\mathcal{X}_{\{y_1\}})\) are distinct. If \(Fq-b-cl(\mathcal{X}_\lambda) = \tilde{1}_X\) then \(\mathcal{X}_\delta\) is fuzzy q-b open, hence \(\mathcal{X}_{\{y_1\}}\) is fuzzy q-b closed, which mean that \(Fq-b-cl(\mathcal{X}_{\{y_1\}}) = \mathcal{X}_{\{y_1\}}\) which is not equal to \(Fq-b-cl(\mathcal{X}_{\{y_1\}})\).

Theorem 7.5: In any fuzzy quad topological space \((X, \tau_1, \tau_2, \tau_3, \tau_4)\), if distinct fuzzy points have distinct fuzzy q-b closure then \(\mathcal{X}_\lambda\) is \(Fq-b-T_0\) space.
Proof: Let $\mathcal{X}_{(x_1)}$, $\mathcal{X}_{(y_1)} \leq \mathcal{X}$ with $\mathcal{X}_{(x_1)} = \mathcal{X}_{(y_1)}$, with $Fq - cl - b(\mathcal{X}_{(y_1)})$ is not equal to $Fq - b - cl(\mathcal{X}_{(x_1)})$, hence there exists $\mathcal{X}_{(z)} \leq \mathcal{X}$ such that $\mathcal{X}_{(z)} \leq Fq - b - cl(\mathcal{X}_{(x_1)})$ but $\mathcal{X}_{(z)} > Fq - b - cl(\mathcal{X}_{(y_1)})$ or $\mathcal{X}_{(z)} \leq Fq - b - cl(\mathcal{X}_{(x_1)})$ but now without loss of generality, let $\mathcal{X}_{(z)} \leq Fq - b - cl(\mathcal{X}_{(x_1)})$

But $\mathcal{X}_{(z)} > Fq - b - cl(\mathcal{X}_{(x_1)})$. If $\mathcal{X}_{(z)} \leq Fq - b - cl(\mathcal{X}_{(y_1)})$, then $Fq - b - cl(\mathcal{X}_{(x_1)})$ is contained in $Fq - b - cl(\mathcal{X}_{(y_1)})$, hence $\mathcal{X}_{(z)} \leq Fq - cl(\mathcal{X}_{(y_1)})$, which is a contradiction, this mean that $\mathcal{X}_{(z)} > Fq - b - cl(\mathcal{X}_{(y_1)})$ hence $\mathcal{X}_{(z)} > Fq - b - cl(\mathcal{X}_{(y_1)}) \mathcal{X}_{(x_1)} \leq Fq - cl - b(\mathcal{X}_{(y_1)})$, hence $X$ is fuzzy q-b - $T_0$ space.

Definition 7.6: A fuzzy quad topological space $X$ is said to be fuzzy q-b-$T_i$ space if and only if for any given pair of distinct fuzzy points $\mathcal{X}_{(x_1)}$, $\mathcal{X}_{(y_1)}$ of $X$ there exist two fuzzy q-b-open sets $\mathcal{X}_\lambda$, $\mathcal{X}_\delta$ such that $\mathcal{X}_{(x_1)} \leq \mathcal{X}_\lambda$, $\mathcal{X}_{(y_1)} > \mathcal{X}_\lambda$ and $\mathcal{X}_{(y_1)} \leq \mathcal{X}_\delta$. $\mathcal{X}_{(x_1)} \geq \mathcal{X}_\delta$.

Theorem 7.7: Every fuzzy q-b-$T_i$ space is a fuzzy q-b-$T_0$ space.

Definition 7.8: A fuzzy quad topological space $X$ is said to be fuzzy q-b-$T_2$ space if and only if for $\mathcal{X}_{(x_1)}$, $\mathcal{X}_{(y_1)} \leq \tilde{1}_X$, $\mathcal{X}_{(x_1)} \neq \mathcal{X}_{(y_1)}$ there exist two disjoint fuzzy q-b-open sets $\mathcal{X}_\lambda$, $\mathcal{X}_\delta$ in $X$ such that $\mathcal{X}_{(x_1)} \leq \mathcal{X}_\lambda$, $\mathcal{X}_{(y_1)} \leq \mathcal{X}_\delta$.

Theorem 7.9: Every fuzzy q-b-$T_2$ space is fuzzy q-b-$T_1$ space.

Proof: Let $X$ is a fuzzy q-b-$T_2$ space and let $\mathcal{X}_{(x_1)}$, $\mathcal{X}_{(y_1)}$ in $X$ with $\mathcal{X}_{(x_1)} \neq \mathcal{X}_{(y_1)}$, so by hypothesis there exist two disjoint fuzzy q-b open, say $\mathcal{X}_\lambda$, $\mathcal{X}_\delta$ such that $\mathcal{X}_{(x_1)} \leq \mathcal{X}_\lambda$, $\mathcal{X}_{(y_1)} \leq \mathcal{X}_\delta$ but $\mathcal{X}_\lambda \land \mathcal{X}_\delta = \tilde{0}_X$ hence $\mathcal{X}_{(x_1)} \geq \mathcal{X}_\delta$, $\mathcal{X}_{(y_1)} \geq \mathcal{X}_\lambda$ i.e. $X$ is fuzzy q-b-$T_1$ space.

Theorem 7.10: Every fuzzy q-b-$T_1$ space is afuzzy q-b-$T_2$ space.

Proof: Let $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ be a fuzzy q-b-$T_3$ space and let $\mathcal{X}_{(x_1)}$, $\mathcal{X}_{(y_1)}$ be two distinct fuzzy points of $X$. Now by definition, $X$ is also a fuzzy q-b-$T_1$ space & so $\mathcal{X}_{(x_1)}$ is a fuzzy q-closed set. Also $\mathcal{X}_{(y_1)} \geq \mathcal{X}_{(x_1)}$. Since $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ is a fuzzy q-b regular space, there exist fuzzy q-b open sets $\mathcal{X}_\lambda$ and $\mathcal{X}_\delta$ such that $\mathcal{X}_{(x_1)} < \mathcal{X}_\lambda$, $\mathcal{X}_{(y_1)} < \mathcal{X}_\delta$ and $\mathcal{X}_\lambda \land \mathcal{X}_\delta = \tilde{0}_X$. Also $\mathcal{X}_{(x_1)} < \mathcal{X}_\lambda \Rightarrow \mathcal{X}_{(x_1)} \leq \mathcal{X}_\lambda$ Thus $\mathcal{X}_{(x_1)}$, $\mathcal{X}_{(y_1)}$ belong respectively to disjoint fuzzy q-b open sets $\mathcal{X}_\lambda$ and $\mathcal{X}_\delta$. According $(X, \tau_1, \tau_2, \tau_3, \tau_4)$ is a $Fq - b - T_2$ space.
VII. CONCLUSION

In this paper the idea of fuzzy q–b open sets, fuzzy q–b separation in fuzzy quad topological spaces were introduced and fuzzy q-b continuity were studied.

REFERENCES