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# A Study on the Numerical Accuracy and Efficiency of the Bisection Method in Finding Square Roots of Positive Real Numbers

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***Abstract—***Calculating square roots of positive real numbers play a vital role in scientific and engineering computing. From the daily-life used calculators to the computer software used as a calculator often use the square root function. However, finding the real roots of algebraic and transcendental equations is one of the most exciting topics in numerical computation. There are several methods, such as the Bisection, Secant, Iteration, and Newton-Raphson's methods, to do that. The speciality of the Bisection method is its robustness and needs no stability criterion. Although it convergences slowly, it always convergences. In this study, we assessed the numerical accuracy through the root mean square error (RMSE) value for this method by applying it to finding square roots of some positive real numbers. Also, we calculated the computational time and number of iterations to convergence to an exact root with the error tolerance of 0.000001 for assessing the method's efficiency. The method's RMSE value obtained in our study is of order $10^{-7}$, indicating its reasonably acceptable accuracy level. We got this accuracy within 23 iterations in each case, and the computational time is a tiny fraction of a millisecond; these indicate the excellent efficiency level of the method. Our inquiry has found the method reasonably acceptable, efficient, and robust.

***Keywords—***Bisection method; Square root; Algebraic and transcendental equations; Numerical computation of zeros; Root mean square error; Accuracy; Rate of convergence; Efficiency

## I. Introduction

Numerical analysis is the combination of theoretical and computational inquiry into computer solutions to mathematical problems [1]. It is the study of numerical data computation techniques [1]. In many cases, this requires constructing a series of approximations[2-5]; hence, the issues include the pace of convergence, the accuracy of the response, and the efficacy of the numerical methods [6]. The primary areas of theoretical interest in numerical analysis are global or local error bounds and methods' stability and convergence rates [5, 6]. The purpose of numerical analysis is to develop effective numerical methods for approximating solutions to many mathematical problems and examining their accuracy and stability characteristics [1]. Thus, numerical analysis is essential in every discipline of research and engineering.

Numerical methods have been used in electrical engineerings, such as system theory, automatic control, and electrical equipment design. Numerical methods are important to solve complicated heat transfer problems. It is important to Data Scientists. In complicated geometry of structural engineering problems, numerical methods give the most acceptable solution to design the figure. These methods are useful for determining the adequate population size and locating and computing periodic orbits in the molecular system. One can refer to [7-15].

It should be mentioned that calculating the square roots of positive real numbers is essential in scientific and engineering computing and many other fields. The square root function is often used in various applications, ranging from everyday calculators to computer software used as a calculator. Finding real roots of algebraic and transcendental equations, on the other hand, is one of the fascinating issues in numerical computing. There are a variety of methods for doing this, including the Bisection, Secant, Iteration, and Newton-Raphson's methods [16, 17]. The Bisection method is distinguished by its robustness, which eliminates the requirement for a stability condition. Also, the method might serve as a baseline for other methods like Newton-Raphson's method.

In addition, the Bisection method has been used in various fields of mathematical, statistical, and engineering sciences. Alpaslam Ersoz and Mehmet Kurban [7] used bisection method successfully to implement on an electrical circuit element. Shengwen Xu, Xuefeng Wang and Lei Wang [8] propose the use of the bisection method to get the efficiency of the Dynamic-Capability analysis, which is expected to be improved. Vishai V Mehte and Durgesh Chandrakar [9] also suggested implying the bisection method in electrical circuits. The author in [12] implemented the method to solve the multidimensional problem in a fashion extending the typical case. Some more studies can be found in [18-25].

However, it can be noted that the Bisection method always convergences although the method is sluggish to convergence. Besides convergency, accuracy is also an important property for a numerical method [1].

In this study, therefore, we examined the numerical accuracy of the Bisection method by calculating the root mean square error (RMSE) value when it was used to determine the square roots of certain positive real numbers. Additionally, we determined the computing time and the number of iterations necessary to reach an exact root in order to evaluate the method's efficiency. Our investigation found the method reasonably acceptable, efficient, and robust.

The rest of the paper is organized as follows. Section II discusses the Bisection method and its features. Section III presents the Algorithm of the Bisection method. Then, Section IV deals with the simulation outcomes. Finally, the conclusion and future scope are presented in Section V.

## II. THE BISECTION METHOD AND ITS FEATURES

The Bisection method [16, 17] requires repetitive halves of subintervals of $[a,b]$ and identifies the subinterval that contains the root at each step. This method is most effective when we have merely an interval containing the root. It also works when there are several roots in the interval. However, we consider that the root of this problem is unique.

The Bisection method is based on the following theorem: "If a function $f(x)$ is continuous between $a$ and $b$, and $f(a).f(b)<0$ then there exist at least one root between $a$ and $b$."

Let, $f(a).f(b)<0$. Then, the first approximation is $x_0=(a+b)/2$. In this case, we have the following two possibilities: (i) $f(x_0)=0$ and (ii) $f(x_0)\neq 0$.

Now, if case (i) occurs, then $x_0$ is the root at the equation $f(x)=0$, and the process is terminated. If case (ii) occurs, then the root either lies between $x_0$ and $a$ or between $x_0$ and $b$ according to as $f(x_0)f(a)<0$ or $f(x_0)f(b)<0$. So, we have the second approximation,

$$x_1=\frac{x_0+a}{2} \text{ or } x_1=\frac{x_0+b}{2}.$$

Similarly, the process of the Bisection method is repeated until the root is obtained with the desired accuracy.

There are two significant benefits to this method. The first is its extreme robustness. In a certain number of iterations, it is guaranteed to obtain an approximated root within a specified level of accuracy. It should be noted that, the solution falls inside an interval of size $|b-a|$ at the beginning of the process. If the solution is not found, in the second step, it falls inside an interval of size $(|b-a|)/2$. Finally, the solution must fall inside an interval of size $\left|b^{(n)}-a^{(n)}\right|=(|b-a|)/2^{n-1}$ at the n[th] iteration. Thus, if the tolerance for inaccuracy is $\delta$, then we can ensure that $n=\log_2((b-a)/\delta)+1$ iterations will fall under the tolerance. The method's avoidance of dependency on the function's derivatives is another significant advantage. As a result, it may be used to locate the roots of functions that are not smooth. While an advantage in certain circumstances, the Bisection method is slower than other methods that employ the function's curvature to find the root.

## III. ALGORITHM

The Bisection method's algorithm is offered for use in developing a computer program in a suitable programming language. The following is the algorithm:

*Algorithm 1:* The algorithm for the Bisection method is given below.

**Step 1** INPUT:: interval's endpoints a, b; tolerance tol.
**Step 2** IF (f(a) == 0) THEN OUTPUT ("The root is a") & STOP.
IF (f(b) == 0) THEN OUTPUT ("The root is b") & STOP.
IF (f(a)*f(b) < 0) THEN GO TO **Step 3**.
IF (f(a)*f(b) > 0) THEN OUTPUT ("There is no real root in the interval [a, b]") & STOP.
**Step 3** set i = 1;
ar(i) = (a + b)/2; *%ar is approximated root*
IF (f(ar(i)) == 0) THEN GO TO **Step 9**.
IF (f(ar(i))*f(a) < 0) THEN a = ar(i);
ELSE SET b = ar(i).
**Step 4** SET i = i + 1;
ar(i) = (a + b)/2;
IF (f(ar(i)) == 0) THEN GO TO **Step 9**.
**Step 5** WHILE (|ar(i) – ar(i – 1)| > tol) DO **Steps 6-8**.
**Step 6** IF(f(ar(i))*f(a) < 0) THEN a = ar(i);
ELSE SET b = ar(i).
**Step 7** SET i = i + 1;
ar(i) = (a + b)/2;
**Step 8** IF (f(ar(i) == 0) THEN GO TO **Step 9**.
**Step 9** OUTPUT ("The root is ar(i)");
STOP.

We have developed a MATLAB program using the given algorithm for computer implementation. Then, the program was implemented to obtain the approximated square roots of some positive real numbers to test their accuracy and the method's efficiency. In the next section, the process of implementation of the MATLAB program is detailed.

 

## IV. RESULTS AND DISCUSSION

The simulation was run using the MATLAB software on a computer with the Windows 8.1 Pro 64-bit operating system with eight gigabytes RAM and the Intel Core i5-4570 CPU @ 3.20 GHz hardware configuration. We have considered the following function for implementing the MATLAB program to test the accuracy of the approximated real square roots of the positive real numbers 1, 4, 9, 16, and 25, obtained by the Bisection method.

$$f(x) = x^2 - n, \tag{1}$$

where $n$=1,4,9,16, and 25. The methods efficiency was tested by the computing time and the number of iterations necessary to reach the corresponding exact root of the following equation.

$$x^2 - n = 0, \tag{2}$$

where $n$=1,4,9,16, and 25.

We considered the interval [0, 7] because the square roots of the adopted numbers are in the interval. Also, the approximated roots were calculated with the error tolerance of 0.000001. The values of the square roots of the adopted numbers obtained by the Bisection method after each iteration have been shown in Figs. 1-5.

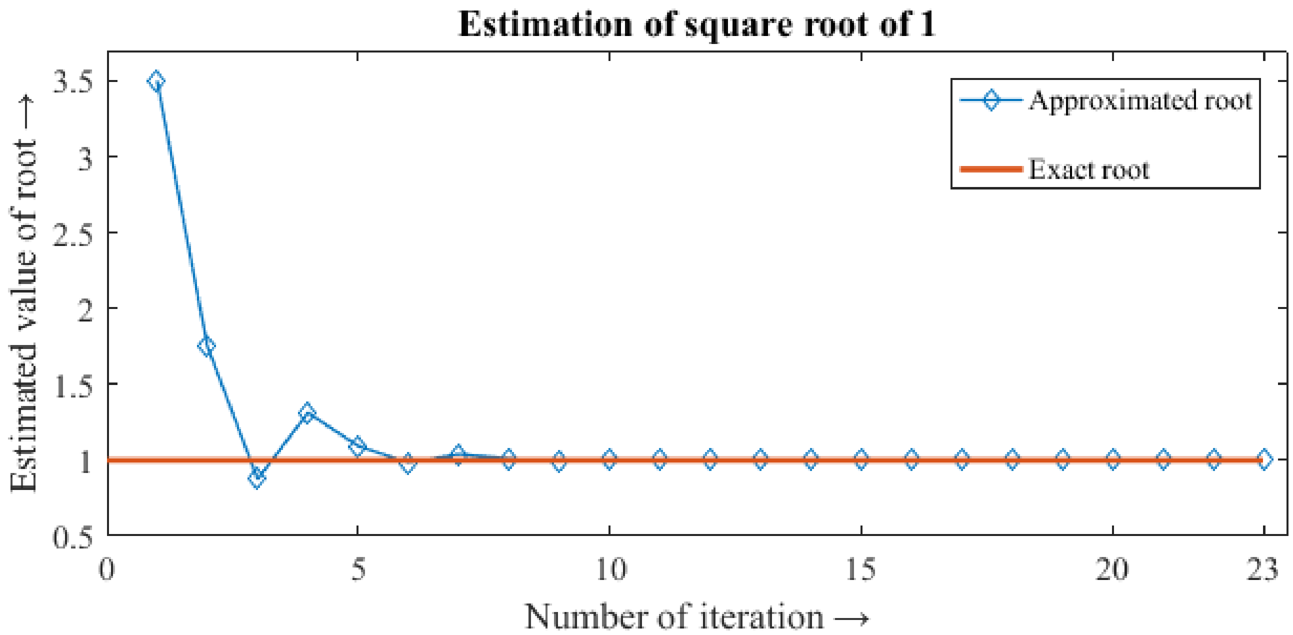


Figure 1. Estimated values of the square root of 1 after each iteration.

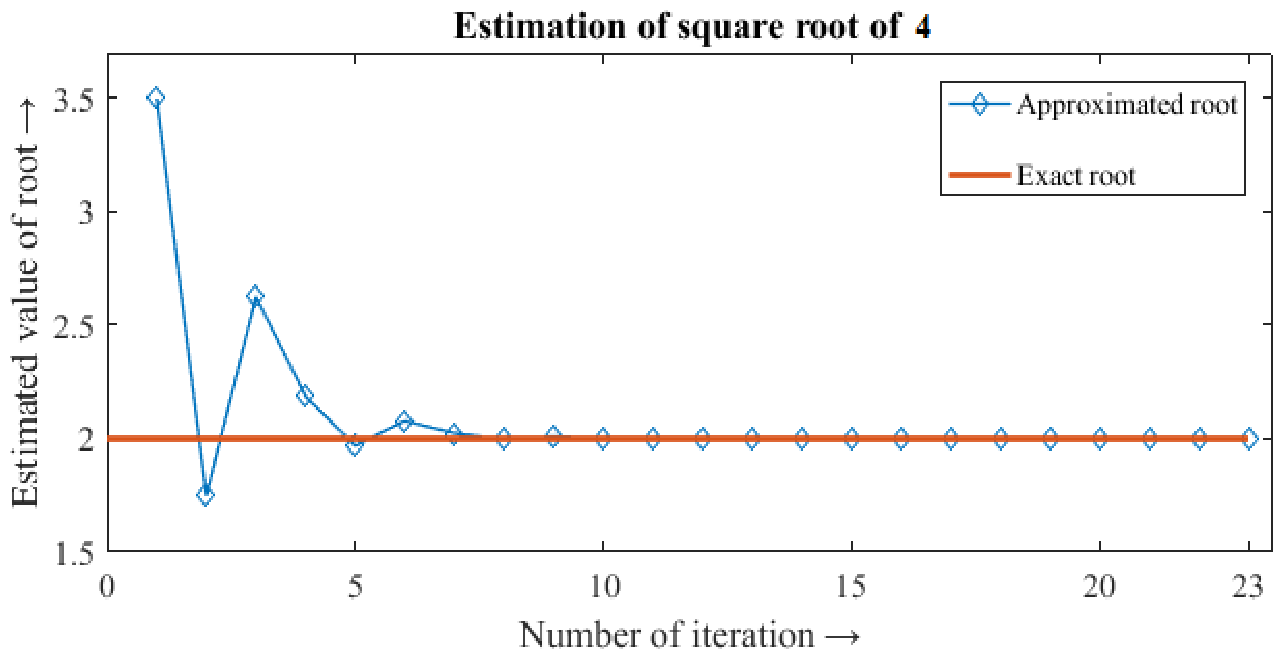


Figure 2. Estimated values of the square root of 4 after each iteration.

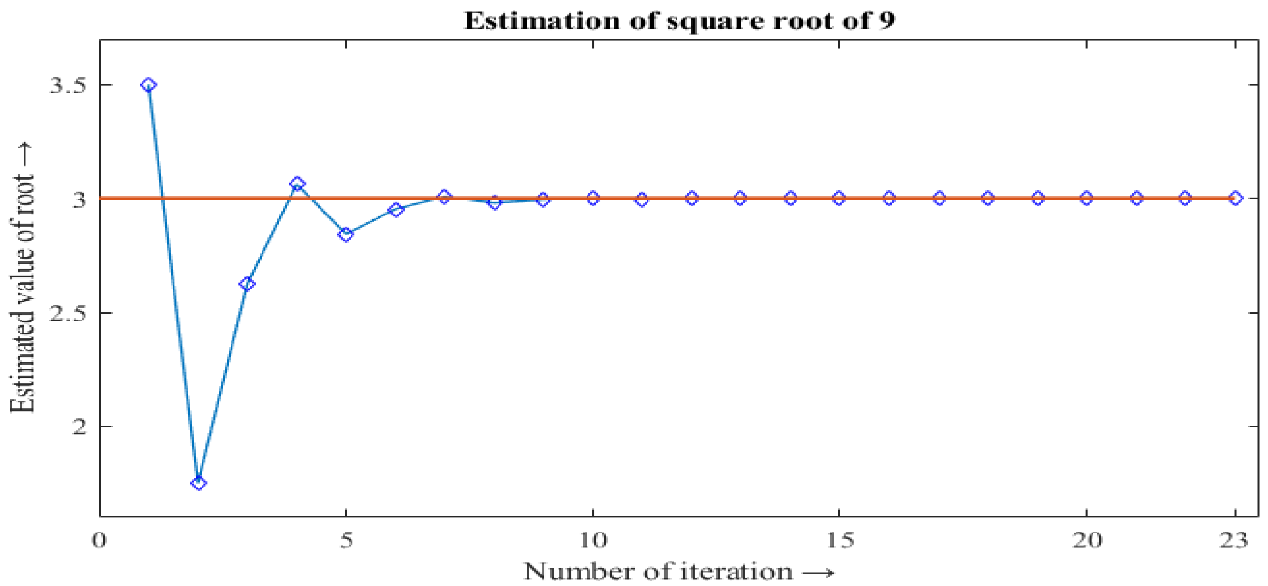


Figure 3. Estimated values of the square root of 9 after each iteration.

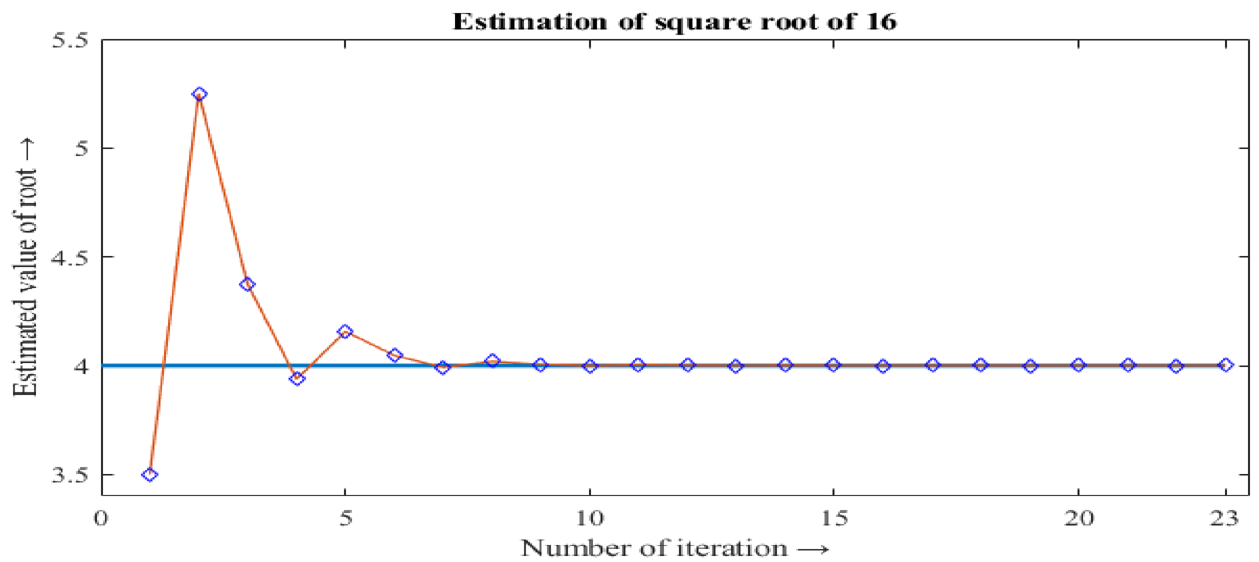


Figure 4. Estimated values of the square root of 16 after each iteration.

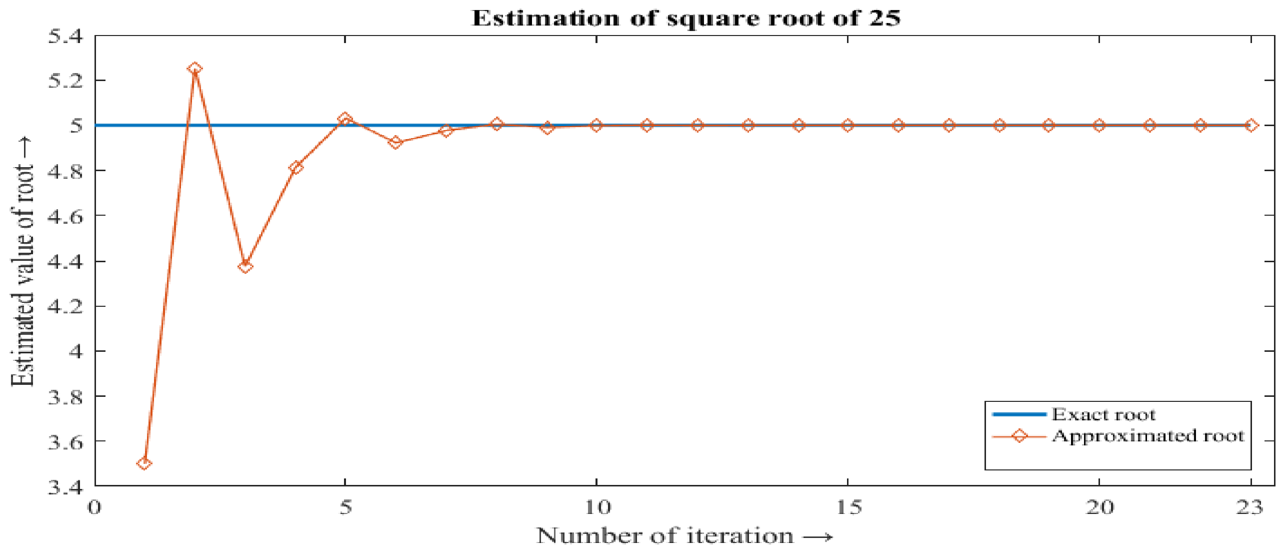


Figure 5. Estimated values of the square root of 25 after each iteration.

 

Figures 1-5 show that the Bisection method requires 23 iterations for estimating the square root with the desired error tolerance in each case. Table 1 presents the exact and approximated values of the roots, errors between them, and elapsed time in calculating the square roots of 1, 4, 9, 16, and 25 by the Bisection method.

Table 1. The exact and approximated values, errors between them, and elapsed time in calculation of the square roots of 1, 4, 9, 16, and 25 by the Bisection method

| Equation | The exact value of the root | Approximated value of the root | Error | Time (ms) |
|---|---|---|---|---|
| $x^2-1=0$ | 1 | 1.0000003576 | -3.5763e-07 | 0.0055 |
| $x^2-4=0$ | 2 | 1.9999998808 | 1.1921e-07 | 0.0019 |
| $x^2-9=0$ | 3 | 2.9999994040 | 5.9605e-07 | 0.0013 |
| $x^2-16=0$ | 4 | 4.0000005960 | -5.9605e-07 | 0.0013 |
| $x^2-25=0$ | 5 | 5.0000001192 | -1.1921e-07 | 0.0016 |

The last column of Table 1 presents the elapsed time given in milliseconds to calculate the square roots of the mentioned numbers. It is found that the elapsed times are minimal, which can be neglected. Thus, the efficiency of the Bisection method can be considered very high in terms of the elapsed time and number of iterations that need convergence to an exact root.

Moreover, the RMSE value is calculated, following [1], with the errors presented in the second last column of Table 1. It is to be noted here that the accuracy of the approximated solutions obtained by any numerical method is inversely proportional to the RMSE value [1]. The calculated RMSE value for the Bisection method to compute the square roots of 1, 4, 9, 16, and 25 with six decimal places accuracy is $4.16380147\,852253 \times 10^{-7}$, which is of order $10^{-7}$. The small order of the RMSE value indicates the Bisection method's accuracy level, which is acceptable in scientific computing.

In this work, we have tested the accuracy and efficiency, two essential criteria for studying numerical methods, of the Bisection method in finding square roots of positive real numbers. The method is found effective in calculating square roots of positive real numbers.

## V. CONCLUSION AND FUTURE SCOPE

The robustness, efficiency, and numerical accuracy of the Bisection method, the simplest and common one among the existing root-finding methods, in finding square roots of positive real numbers have been examined. We assessed the numerical accuracy through the RMSE value for the method. Also, we calculated the computational time and number of iterations to convergence to an exact root with the error tolerance of 0.000001 for assessing the method's efficiency. The found RMSE value indicates the method's fairly acceptable accuracy level. In addition, the iteration numbers and the elapsed time indicate the excellent efficiency level of the method. Our inquiry has found the method fairly acceptable, efficient, and robust.

Further, the robustness, efficiency, and numerical accuracy of the Bisection method can be tested in finding real $n^{\text{th}}$ roots of positive real numbers because these $n^{\text{th}}$ root finding functions are used less or more in scientific and engineering computing.

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