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Weak Compatible and Reciprocally Continuous Maps in Non-Archimedean Menger PM-Space

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Abstract-In this paper, we introduce weak-compatible maps and reciprocally continuous maps in weak non-Archimedean PM-spaces and establish a common fixed point theorem for such maps. Our result generalizes several fixed point theorems in the sense that all maps involved in the theorem can be discontinuous even at the common fixed point.

Keywords and Phrases: *Non-Archimedean Menger probabilistic metric space, Common fixed points, Compatible maps, Weakly Compatible, reciprocally continuous maps.*

AMS Subject Classification (2000). Primary 47H10, Secondary 54H25

I. INTRODUCTION

In this section, we give a short survey of the study of finding weaker forms of commutativity to have a common fixed point. In fact, this problem seems to be of vital interest and was initiated by Jungck [5] with thein troduction of the concept of commuting maps. In 1982, Sessa [11] introduced the notion of weakly commutativity as a generalization of commutativity and this was a turning point in the development of Fixed Point Theory and its

applications in various branches of mathematical sciences. To be precise, Sessa [11] defined the concept of weakly commuting by calling self maps maps A and B of a metric space(X, d) a weakly commuting pair if and only if

$$d(ABx, BAx) \leq d(Ax, Bx),$$

for all x Є X .Further to this other authers gave some common fixed point theorems for weakly commuting maps [1,3, 4]. Note that commuting maps are weakly commuting, but the converse is not true .

In 1986, Jungck [5] introduced the new notion of compatibility of maps as a generalization of weak commutativity.Thereafter, a flood of common fixed point theorems was produced by using the improved notion of compatibility of maps. Later on, Jungck [5] introduced the concept of compatible maps of type (A) or of type (α), Pathak et al.[2] introduced the compatible maps of type (B) or of type (ß), type (C) and type (P) in metric spaces and using these concepts, several researchers and mathematicians have proved common fixed point theorems. Recently, Cho et.al, [1] introduced the notion of compatible maps of type (A) in non-Archimedean Menger PM-spaces and proved some interesting results. In this direction, a weaker notion of compatible maps, called semi-compatible maps, was introduced in fuzzy metric spaces by Singh et. al. [12,13]. In particular, they proved that the concept of semi-compatible maps is equivalent to the concept of compatible maps and compatible maps of type (a) and of type (ß) under some conditions on the maps.

In this paper, attempts have been made to introduce weak-compatible and reciprocally continuous maps in weak non-Archimedean Menger PM-spaces Here, we also present the concepts of compatible maps of type (A - 1) and (A-2) Afterwards, Jain et. al. [6,7] proved the fixed point theorem using the concept of weak compatible maps in Menger space.

II. PRELIMINARIES

Definition 1. A distribution function is a fction ,F: $[-\infty,\infty] \rightarrow [0,1]$ which is left continuous on R, non decreasing and $F(-\infty) = 0$, $F(+\infty) = 1$.If X is non empty set, $F: X \times X \rightarrow \Delta$ is called a probabilistic distance on X and F(x,y) is usually denoted by F_{xy}.

Definition 2. Let X be a non-empty set and D be the set of all left-continuous distribution functions. An ordered pair (X, F) is called a non-Archimedean probabilistic metric space (shortly a N.A. PM-space) if F is a mapping from X×X×X into D satisfying the following conditions (the distribution function F(u,v,w) is denoted by $F_{u,v,w}$ for all u, v,w X) :

(PM-1) $F_{u,v,w}(x) = 1$, for all x >0, if and only if at least two of the three points are equal;

(PM-2) $F_{u,v,w} = F_{u,w,v} = F_{w,v,u}$;

(PM-3) $F_{u,v,w}(0) = 0$;

(PM-4) If $F_{u,v,s}(x_1) = 1$, $F_{u,s,w}(x_2) = 1$ and $F_{s,v,w}(x_3) = 1$

then $F_{u,v,w}(\max\{x_1, x_2, x_3\}) = 1$

for all u, v, w, s X and $x_1, x_2, x_3 \geq 0$.

Definition 3. A t-norm is a function $\Delta : [0,1]\times[0,1]\times[0,1] \to [0,1]$ which is associative, commutative, non-decreasing in each coordinate and $\Delta(a,1,1) = a$ for every a [0,1].

Definition 4. A N.A. Menger PM-space is an ordered triple (X, F,Δ), where (X, F) is a non-Archimedean PM-space and Δ is a t-norm satisfying the following condition:

(PM-5) $F_{u,v,w}(\max\{x,y,z\}) \geq \Delta(F_{u,v,s}(x), F_{u,s,w}(y)), F_{s,v,w}(y)$

For all u,v,w,s X and $x,y,z \geq 0$

If the triangular inequality (PM-5) is replaced by the following

(WNA) Fu,v,w (x) = max{Fu,v,s (x) , Fu,s,w (x/2) * Fu,s,w (x) Fs,v,w

(x/2) * Fs,v,w (x), Fu,v,s , (x/2)}, for all x, y, z X and t > 0,

then the triple (X, F, *) is called a weak non-Archimedean Menger probabilistic metric space (shortly Menger WNAPM-space). Obviously every Menger NAPM-space is itself a Menger WNA-space (see Vetro for the same concept in fuzzy metric spaces).

Remark 1. Condition (WNA) does not imply that Fu,v,w (x) is nondecreasing and thus a Menger WNAPM- space is not necessarily a Menger PM-space. If F u,v,w(x) is nondecreasing, then a Menger WNA-space is a Menger PM-space.

Remark 2. Recall that a Menger space is also a fuzzy metric space, for more details see Hadzic [14].

Example 1. Let X = [0, + ∞), a * b = ab for every a, b [0, 1]. Define Fx,y(t) by: Fx,y (0) = 0, Fx,x (t) = 1 for all t > 0, Fx, y (t) = t for x ≠ y and 0 < t ≤ 1, Fx,y (t) = t/2 for x ≠ y and 1 < t ≤ 2, Fx,y (t) = 1 for x ≠ y and t > 2.

Then (X, F , *) is a Menger WNAPM-space, but it is not a PM-space.

We recall that the concept of neighborhood in Menger PM-spaces was introduced by Schweizer and Sklar [10] as follows;

If x X, ε > 0 and (0, 1), then an (ε ,)-neighborhood of x, Ux (ε ,) is defined by

Ux (ε,) = {y X : Fx,y (ε) > 1 - }.

If the t-norm * is continuous and strictly increasing, then (X, F , *) is a Hausdorff space in the topology induced by the family {Ux (ε,) : x X, ε > 0, (0, 1)} of neighborhoods.

Let Ω = g{ such that : [0, 1] → [0, + ∞) is continuous, strictly decreasing, g(1) = 0 and g(0) < + ∞}.

Definition 5. A Menger WNAPM-space (X, F, Δ) is said to be of type $(C)_g$ if there exists a g Ω such that

$$g(F_{x,y,z}(t)) \leq g(F_{x,y,a}(t)) + g(F_{x,a,z}(t)) + g(F_{a,y,z}(t))$$

for all x, y, z, a X and t ≥ 0, where Ω = {g | g : [0,1] → (0,∞) is continuous, strictly decreasing, g(1) = 0 and g(0) < ∞ }.

Definition 6. A Menger WNAPM-space (X, F, *) is said to be type $(D)_g$ if there exists a g Ω such that

$$g(\Delta(t_1 * t_2 * t_3) \leq g(t_1) + g(t_2) + g(t_3)$$

for all t_1, t_2, t_3 [0,1].

Remark 3.

1. If a weak WNA Menger PM-space (X, F, *) is of type $(D)_g$ then (X, F, Δ) is of type $(C)_g$.On the other hand if (X, F, *) is a WNAPM-space such that $a * b \geq \max[a+ b - 1,0]$ for all a,b [0,1], then (X, F, *) is of type $(D)_g$ For g Ω defined by $g(t) = 1- t$, $t \geq 0$

Throughout this paper, even when not specified (X, F, *) will be a complete Menger WNAPM-space of type $(D)_g$ with a continuous strictly increasing t-norm Δ.

Let : [0, +∞) → [0, ∞) be a function satisfied the condition (Φ) :

(Φ) is upper-semicontinuous from the right and (t) < t for all t > 0.

Lemma 1. If a function □□ : $[0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ), then we have

(1) For all $t \geq 0$, $\lim_{n\to\infty}$ □$^n(t) = 0$, □$^n(t)$ is n-th iteration of □□(t).

(2) If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq$ □ (t_n), $n = 1, 2, \ldots$ then $\lim_{n\to\infty} t_n = 0$.

In particular, if $t \leq$ □ (t) for all $t > 0$, then $t = 0$.

Definition 7. Let $A, S : X \to X$ be mappings. A and S are said to be compatible if $g(F_{ASx_n, SAx_n}(t)) = 0$ for all $t > 0$, a □ X, whenever $\{x_n\}$ is a sequence in X such that

$Ax_n = Sx_n = z$ for some z in X .

The notion of reciprocal continuity was defined by Pant [15] in ordinary metric space. Now, following the same line, we introduce reciprocally continuous maps in Menger WNAPM-spaces.

Definition 8. A pair of self-maps (A,S) of a Menger WNAPM-space (X, F, *) is said to be reciprocally continuous if $g(F_{ASx_n,Az,}(t))$ □ 0 and $g(F_{SAx_n,Sz}(t))$ □ 0 for all $t > 0$, whenever there exists a sequence $\{x_n\}$ in X such that Ax_n □ z , Sx_n □ z for some z in X as n □□□□.

If A and S both are continuous, then, they are obviously reciprocally continuous but the converse generally is not true .

Proposition 1. Let A and S be two self-maps of a Menger WNAPM-space (X, F, *). Assume that (A, S) is compatible and reciprocally continuous, then (A,S) is weakly compatible .

Proof. Let $\{ x_n \}$ be a sequence in X such that $A_{x_n} \to z$ and $S_{x_n} \to z$ since the pair of maps (A, S) is reciprocally continuous, then for all $t > 0$, we have $\lim_{n\to\infty} g(F_{ASx_n}, {}_{Az,}(t)) = 0$ and $\lim_{n\to\infty} g(F_{SAx_n}, {}_{Sz}(t)) = 0$

suppose that (A, S) is compatible and reciprocally continuous, then, for $t > 0$, have $\lim_{n\to\infty} g(F_{ASx_n}, {}_{SAx_n}(t)) = 0$, for all x_n □ X

Then we get

$g(F_{ASx_n}, {}_{Sz}(t)) \leq g(F_{ASx_n}, {}_{SAx_n}(t))$

$+ g (F_{SAx_n}, {}_{Sz} (t))$,

And so letting $n \to +\infty$,we obtain

$\lim_{n\to\infty} g(F_{ASx_n}, {}_{Sz}(t))=0$ thus ,A and S are weak compatible.

Naturally, we can define the concept of compatible mappings of type (A-1) and type (A-1) in Menger WNAPM-space is as follows.

Definition 9. Two self-maps A and B of a Menger WNAPM-space (X, F, *) are said to be compatible of type (A-1) if, for all $t > 0$, $\lim_{n□+□□} g(F_{ABx_n,BBx_n}(t)) = 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z □ X as $n \to +\infty$.

Definition 10. Two self-maps A and B of a Menger WNAPM-spac (X , F, *) are said to be compatible of type (A-2) if, for all $t > 0$, $\lim_{n□+□} g(F_{BAx_n,AAx_n}(t)) = 0$, whenever $\{x_n\}$ is a sequence in X such that

Ax_n, Bx_n □ z for some z □ X .

In the following proposition, it is shown that the concept of compatible maps of type (A-1), type (A-2) and if A and B compatible maps of type (A) then the pair (A,B) is compatible of type (A-1) as well as type (A-2).

Proposition 2. Let A and B be two self-maps of a Menger WNAPM-space (X, F, *). If (A,B) are compatible of type A then they are weakly compatible.

Proof. To prove let $\{ x_n \}$ be a sequence in X such that Ax_n , $Bx_n \to z$ for some z in X, as $n \to +\infty$ and let the pair (A, B) be compatible of type (A). we have

$\lim_{n\to\infty} g(F_{ABx_n}, {}_{BBx_n}(t)) = 0$ and

$\lim_{n\to\infty} g(F_{BAx_n}, {}_{AAx_n}(t)) = 0$ (3)

$g(F_{ABx_n}, {}_{BAx_n}(t)) \leq g(F_{ABx_n,BBx_n}(t)) + g(F_{BBx_n}, {}_{BAx_n}(t))$,

letting $n \to +\infty$, we have

$g(F_{ABx_n}, {}_{BAx_n}(t))$ g = 0 for all $t > 0$

Thus $ABx_n = BAx_n$ and we get (A, B) is weakly compatible .Using similar arguments as above, the reader can easily prove the following result.

Proposition 3. Let A and B be two self-maps of a Menger WNAPM-space. If the pair (A,B) is weakly-compatible and reciprocally continuous and $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some $z \in X$ as $n \to +\infty$, then ABz = BAz.

Proof. Suppose $\{x_n\}$ is a sequence in X defined by

$x_n = z$, n = 1, 2, ... and Az = Bz.

Then we have,

$Ax_n, Bx_n \to Bz$ as $n \to \infty$

Since (A, B) is weakly compatible , by triangle inequality

$g(F_{TBx_n, BBx_n}(t)) \leq g(F_{TBx_n, Bz}(t)) + g(F_{Bz, BBx_n}(t))$

Since $BAx_n \to Az$ and $AAx_n \to Az$ as $n \to \infty$

then $g(F_{TBx_n, Bz}(t)) = 0$ and $g(F_{Bz, BBx_n}(t)) = 0$

$\Rightarrow$ $g(F_{TBx_n, BBx_n}(t)) = 0$

$\Rightarrow$ $g(F_{TBz, BBz}(t)) = 0$

i.e. BAz = AAz. (1)

Similarly, we can have

ABz = BBz. (2)

Hence, by (1) and (2), we have

ABz = BAz = AAz = BBz

Before proving our main theorem, we need the following lemma .

Lemma 2. Let A, B, L, M, S and T be self-maps of a complete Menger WNAPM-space (X, F, *) of type $(D)_g$, satisfying

(i) $L(X) \subset ST(X)$, $M(X) \subset AB(X)$;

(ii) for all $x, y \in X$ and $t > 0$,

$g(F_{Lx,My}(t)) \leq \varphi(\max\{g(F_{ABx,Sty}(t)), g(F_{Lx,ABx}(t)), g(F_{My,Sty}(t)), ½[g(F_{ABx,My}(t)) + g(F_{Lx,Sty}(t))]\})$,

where the function $\varphi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ)

proof: Let $x_0 \in X$ Then the sequence $\{y_n\}$ in X, defined by

$Lx_{2n} = STx_{2n+1} = y_{2n}$ and $ABx_{2n+1} = Mx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2, ..., such that $g(F_{y_n, y_{n+1}}(t)) = 0$ for all $t > 0$ is a Cauchy sequence in X. If is not a Cauchy sequence in X, there exist $\varepsilon > 0$, $t > 0$ and two sequences $\{m_i\}$,$\{n_i\}$ of positive integer such that,

(a) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$
(b) $F_{y_{m_i}, y_{n_i}}(t_0)) < 1 - \varepsilon$ and $F_{y_{m_i - 1}, y_{n_i}}(t_0)) \geq 1 - \varepsilon_0$, i = 1,2,3…

Thus, we have

$g(1-\varepsilon_0) < g(F_{y_{m_i}, y_{n_i}}(t_0)) \leq g(F_{y_{m_i}, y_{m_i - 1}}(t_0)) + g(F_{y_{m_i - 1}, y_{n_i}}(t_0))$

$\leq g(F_{y_{m_i}, y_{m_i - 1}}(t_0)) + g(1-\varepsilon_0)$

And letting $i \to +\infty$, we get

$g(F_{y_{m_i}, y_{n_i}}(t_0)) = g(1-\varepsilon_0)$ (3)

on the other hand, we have

$g(1-\varepsilon_0) < g(F_{y_{m_i}, y_{n_i}}(t_0)) \leq g(F_{y_{m_i}, y_{n_i + 1}}(t_0)) + g(F_{y_{n_i + 1}, y_{n_i}}(t_0))$ (4)

Let us assume that both m_i and n_i are even, By contractive condition (ii),we get

$g(F_{y_{m_i}, y_{n_i + 1}}(t_0)) = g(F_{Lx_{m_i}, Mx_{n_i + 1}}(t_0))$

$\leq \varphi(\max\{g(F_{ABx,Sty}(t)), g(F_{Lx,ABx}(t)), g(F_{My,STy}(t)), ½[g(F_{ABx,My}(t)) + g(F_{Lx,STy}(t))]\})$,

that is

$g(F_{y_{m_i}, y_{n_i + 1}}(t_0)) \leq \varphi(\max\{g(F_{y_{m_i - 1}, y_{n_i}}(t_0)), g(F_{y_{m_i - 1}, y_{m_i}}(t_0)), g(F_{y_{n_i}, y_{n_i + 1}}(t_0)), ½ F_{y_{m_i - 1}, y_{n_i + 1}}(t_0)) + g(F_{y_{n_i}, y_{m_i}}(t_0))]\})$,

putting this values in (6), using (5) and letting $i \to +\infty$, we get

$g(1-\varepsilon_0) \leq \varphi(\max\{g(1-\varepsilon_0), 0, 0\ g(1-\varepsilon_0)\}) = \varphi\{g(1-\varepsilon_0)\} < g(1-\varepsilon_0)$

a contradiction. Hence $\{y_n\}$is a Cauchy sequence in X.

III. Main result

Theorem 1. Let A,B,L,M,S and T be self-maps of a complete Menger WNAPM-space (X, F, *) of type $(D)_g$, satisfying

(1.1) L(X) □ ST(X), M(X) □ AB(X);

(1.2) for all x, y □ X and t > 0,

$g(F_{Lx,My}(t))$ □ □ (max{$g(F_{ABx,Sty}(t))$, $g(F_{Lx,ABx,}(t))$, $g(F_{My,Sty}(t))$,

½ [$g(F_{ABx,My}(t)) + g(F_{Lx,Sty}(t))$]}),

where the function □ : [0,+1) □ [0,+1) satisfies the condition (Φ).

(1.3) AB = BA, ST = TS, LB = BL, MT = TM;

(1.4) the pair (M,ST) is compatible.

If the pair (L,AB) is weakly-compatible and reciprocally continuous, then A, B, L, M, S and T have a unique common fixed point.

Proof Let x_0 □ X

From condition (3.1.1) □□ x_1, x_2 □ X such that

$Lx_1 = STx_2 = y_1$ and $Mx_0 = ABx_1 = y_0$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(1.5) $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$

for n = 0, 1, 2, ...

We prove that $g(F_{y_n,y_{n+1}}(t)) = 0$ for all t > 0.

From (1.4) and (1.5), we have

$g(F_{y_{2n}, y_{2n+1}}(t)) = g(F_{Lx_{2n}, Mx_{2n+1}}(t))$

≤□(max{$g(F_{ABx_{2n}, STx_{2n+1}}(t))$, $g(F_{ABx_{2n}, Lx_{2n}}(t))$,

$g(F_{STx_{2n+1}, Mx_{2n+1}}(t))$,

½($g(F_{ABx_{2n}, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, Lx_{2n}}(t))$)})

= □(max{$g(F_{y_{2n-1},y_{2n}}(t))$, $g(F_{y_{2n-1}, y_{2n}}(t))$, $g(F_{y_{2n}, y_{2n+1}}(t))$,

½($g(F_{y_{2n-1}, y_{2n+1}}(t)) + g(1)$)}) ≤ □(max{$g(F_{y_{2n-1}, y_{2n}}(t))$, $g(F_{y_{2n}, y_{2n+1}}(t))$,

½($g(F_{y_{2n-1}, y_{2n}}(t)) + g(F_{y_{2n}, y_{2n+1}}(t))$)}).

If $g(F_{y_{2n-1}, y_{2n}}(t)) \leq g(F_{y_{2n}, y_{2n+1}}(t))$ for all t > 0, then by (1.4)

$g(F_{y_{2n}, y_{2n+1}}(t)) \leq$ □□□$g(F_{y_{2n}, y_{2n+1}}(t))$),

on applying Lemma 2, we have

$g(F_{y_{2n}, y_{2n+1}}(t)) = 0$ for all t > 0.

Similarly, we have

$g(F_{y_{2n+1}, y_{2n+2}}(t)) = 0$ for all t > 0.

Thus, we have

$g(F_{y_n, y_{n+1}}(t)) = 0$ for all t > 0.

On the other hand, if $g(F_{y_{2n-1},y_{2n}}(t)) \geq g(F_{y_{2n},y_{2n+1}}(t))$, then by (3.1.4), we have

$g(F_{y_{2n},y_{2n+1}}(t)) \leq$ □ ($g(F_{y_{2n-1},y_{2n}}(t))$) for all t > 0.

Similarly, $g(F_{y_{2n+1},y_{2n+2}}(t)) \leq$ □ ($g(F_{y_{2n},y_{2n+1}}(t))$) for all t > 0.

Thus, we have

$g(F_{y_n,y_{n+1}}(t)) \leq$ □($g(F_{y_{n-1},y_n}(t))$) for all t > 0 and n = 1, 2, 3, … .

Therefore, by Lemma 2,

$g(F_{y_n, y_{n+1}}(t)) = 0$ for all t > 0, which implies that $\{y_n\}$ is a Cauchy sequence in X by Lemma 1.

Since (X, **F**, □) is complete, the sequence $\{y_n\}$ converges to a point z □ X. Also its subsequences converges as follows :

(1.6)$\{Mx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$,

(1.7)$\{Lx_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$.

Now since the pair of maps (L, AB)is reciprocally continuous, therefore, we have $g(F_{LABx_{2n}, Lz}(t)) \to 0$ and $g(F_{ABLx_{2n}, ABz}(t)) \to 0$ as $n \to +\infty$.

As (L, AB) is weakly compatible, so by Proposition 3, we have

$L(AB)x_{2n} \to ABz$. That is $ABz = Lz$

Putting $x = ABx_{2n}$ and $y = x_{2n+1}$ for t > 0 in (1.2), we get

$g(F_{LABx_{2n},Mx_{2n+1}}(t)) \leq \phi(\max\{g(F_{ABABx_{2n},STx_{2n+1}}(t)), g(F_{ABABx_{2n}, LABx_{2n}}(t)), g(F_{STx_{2n+1}, Mx_{2n+1}}(t)),$

$\frac{1}{2}(g(F_{ABABx_{2n}, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, LABx_{2n}}(t)))\}).$

Letting $n \to \infty$, we get

$g(F_{ABz,z}(t)) \leq \phi(\max\{g(F_{ABz,z}(t)), g(F_{ABz, ABz}(t)), g(F_{z, z}(t)) \quad \frac{1}{2}(g(F_{ABz, z}(t)) + g(F_{z, ABz}(t)))\}).$

i.e. $g(F_{ABz,z}(t)) \leq \phi(g(F_{ABz,z}(t)))$

which implies that $g(F_{ABz,z}(t)) = 0$ by Lemma 2 and so we have $ABz = z$

Putting $x = z$ and $y = x_{2n+1}$ for $t > 0$ in (1.2), we get

$g(F_{Lz,Mx_{2n+1}}(t)) \leq \phi(\max\{g(F_{ABz,STx_{2n+1}}(t)), g(F_{ABz, Lz}(t)), g(F_{STx_{2n+1}, Mx_{2n+1}}(t)), \frac{1}{2}(g(F_{ABz, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, Lz}(t)))\}).$

Letting $n \to \infty$, we get

$g(F_{Lz,z}(t)) \leq \phi(\max\{g(F_{z,z,}(t)), g(F_{z, Lz}(t)), g(F_{z, z}(t)) \quad \frac{1}{2}(g(F_{z, z}(t)) + g(F_{z, Lz}(t)))\})$

i.e. $g(F_{Lz,z}(t)) \leq \phi(g(F_{Lz,z}(t)))$

which implies that $g(F_{Lz,z}(t)) = 0$ by Lemma 2 and so we have

$Lz = z$.

Therefore, $ABz = Lz = z$.

Putting $x = Bz$ and $y = x_{2n+1}$ for $t > 0$ in (1.2), we get

$g(F_{LBz,Mx_{2n+1}}(t)) \leq \phi(\max\{g(F_{ABBz,STx_{2n+1}}(t)), g(F_{ABBz,LBz}(t)), g(F_{STx_{2n+1},Mx_{2n+1}}(t)), \frac{1}{2}(g(F_{ABBz, Mx_{2n+1}}(t)) + g(F_{STx_{2n+1}, LBz}(t)))\}).$

As $BL = LB$, $AB = BA$, so we have

$L(Bz) = B(Lz) = Bz$ and $AB(Bz) = B(ABz) = Bz$.

Letting $n \to \infty$, we get

$g(F_{Bz, z}(t)) \leq \phi(\max\{g(F_{Bz,z}(t)), g(F_{Bz, Bz}(t)), g(F_{z,z}(t)), \frac{1}{2}(g(F_{Bz, z}(t)) + g(F_{z, Bz}(t)))\})$

i.e $g(F_{Bz, z}(t)) \leq \phi(g(F_{Bz, z}(t)))$

which implies that $g(F_{Bz, z}(t)) = 0$ by Lemma 2 and so we have

$Bz = z$.

Also, $ABz = z$ and so $Az = z$.

Therefore, $Az = Bz = Lz = z$. (1.8)

As $L(X) \subset ST(X)$, there exists $w \in X$ such that $z = Lz = STw$.

Putting $x = x_{2n}$ and $y = w$ for $t > 0$ in (1.2), we get

$g(F_{Lx_{2n},Mw}(t)) \leq \phi(\max\{g(F_{ABx_{2n},STw}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), g(F_{STv, Mw}(t)),$

$\frac{1}{2}(g(F_{ABx_{2n}, Mw}(t)) + g(F_{STw, Lx_{2n}}(t)))\}).$

Letting $n \to \infty$ and using equation (1.7), we get

$g(F_{z,Mw}(t)) \leq \phi(\max\{g(F_{z, z}(t)), g(F_{z, z}(t)), g(F_{z, Mw}(t)), \frac{1}{2}(g(F_{z, Mw}(t)) + g(F_{z, z}(t)))\})$

i.e. $g(F_{z,Mw}(t)) \leq \phi(g(F_{z,Mw}(t)))$

which implies that $g(F_{z,Mw}(t)) = 0$ by Lemma 2 and so we have

$z = Mw$.

Hence, $STw = z = Mw$.

As (M, ST) is weakly compatible, we have

$STMw = MSTw$.

Thus, $STz = Mz$.

Putting $x = x_{2n}$, $y = z$ for $t > 0$ in (1.2), we get

$g(F_{Lx_{2n},Mz}(t)) \leq \phi(\max\{g(F_{ABx_{2n},STz}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), g(F_{STz, Mz}(t)), \frac{1}{2}(g(F_{ABx_{2n}, Mz}(t)) + g(F_{STz, Lx_{2n}}(t)))\}).$

Letting $n \to \infty$ and using equation (3.8) and Step 5, we get

$g(F_{z,Mz}(t)) \leq \phi(\max\{g(F_{z,Mz}(t)), g(F_{z, z,}(t)), g(F_{Mz, Mz}(t)), \frac{1}{2}(g(F_{z, Mz}(t)) + g(F_{Mz, z}(t)))\})$

i.e. $g(F_{z,Mz}(t)) \leq \phi(g(F_{z,Mz}(t)))$

which implies that $g(F_{z,Mz}(t)) = 0$ by Lemma 2 and so we have

$z = Mz$.

Putting $x = x_{2n}$ and $y = Tz$ for $t > 0$ in (1.2), we get

$g(F_{Lx_{2n},MTz}(t)) \leq \phi(\max\{g(F_{ABx_{2n},STTz}(t)), g(F_{ABx_{2n}, Lx_{2n}}(t)), g(F_{STTz, MTz}(t)), ½(g(F_{ABx_{2n}, MTz}(t)) + g(F_{STTz, Lx_{2n}}(t)))\})$.

As MT = TM and ST = TS, we have

MTz = TMz = Tz and ST(Tz) = T(STz) = Tz.

Letting $n \to \infty$ we get

$g(F_{z,Tz}(t)) \leq \phi(\max\{g(F_{z,Tz}(t)), g(F_{z,z}(t)), g(F_{Tz,Tz}(t)),½(g(F_{z,Tz}(t)) + g(F_{Tz,z}(t)))\})$

i.e $g(F_{z,Tz}(t)) \leq \phi(g(F_{z,Tz}(t)))$,

which implies that $g(F_{z,Tz}(t)) = 0$ by Lemma 2 and so we have

z = Tz.

Now STz = Tz = z implies Sz = z.

Hence Sz = Tz = Mz = z. (1.9)

Combining (1.8) and (1.9), we get

Az = Bz = Lz = Mz = Tz = Sz = z.

Hence, the six self maps have a common fixed point z.

(Uniqueness) Let u be another common fixed point of A, B, S, T, L and M; then

Au = Bu = Su = Tu = Lu = Mu = u.

Putting x = z and y = u for t > 0 in (1.2), we get

$g(F_{Lz,Mu}(t)) \leq \phi(\max\{g(F_{ABz,STu}(t)), g(F_{ABz, Lz}(t)), g(F_{STu, Mu}(t)), ½(g(F_{ABz, Mu}(t)) + g(F_{STu, Lz}(t)))\})$.

Letting $n \to \infty$, we get

$g(F_{z, u}(t)) \leq \phi(\max\{g(F_{z, u}(t)), g(F_{z, z}(t)), g(F_{u, u}(t)), ½(g(F_{z, u}(t)) + g(F_{u, z}(t)))\})$

$= \phi(g(F_{z, u}(t)))$,

which implies that $g(F_{z,u}(t)) = 0$ by Lemma 2 and so we have

z = u.

Therefore, z is a unique common fixed point of A, B, S, T, L and M.

This completes the proof.

Remark 1.1. If we take B = T = I, the identity map on X in theorem 1, then the condition (b) is satisfied trivially and we get

Corollary 1.1. Let A, S, L, M : X → X be mappings satisfying the condition:

(a) $L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$;

(b) Either A or L is continuous;

(c) (L, A) is reciprocally continuous and weakly compatible .

(d) $g(F_{Lx,My}(t)) \leq \phi(\max\{g(F_{Ax, Sy}(t)), g(F_{Ax, Lx}(t)), g(F_{Sy, My}(t)), ½(g(F_{Ax, My}(t)) + g(F_{Sy, Lx}(t)))\})$

for all t > 0, where a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfies the condition (Φ).

Then A, S, L and M have a unique common fixed point in X.

Remark 1.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Cho et. al. [2] in the sense that condition of compatibility of the pairs of self maps in a weak non-Archimedean Menger PM-space has been restricted to weak compatible in a weak non-Archimedean Menger PM-space and only one of the mappings of the compatible pair is needed to be reciprocally continuous.

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