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Distributive Lattice Under S-accessibility

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Abstract- If we starts with an acyclic orientation and prohibited flipping and flopping to a simple vertex it ends eventually with an orientation and another orientation is accessible from it then one can show that the set of such orientations of a graph with a given flip-flop can be made it to distributive lattice under s-accessibility. Set of orientations of connected finite graph shows that, any two such orientations having same flow difference around all closed loops has obtained from one another by succession of local moves of a simple type .In this paper simple connected graph with n > 1 vertices has been taken and distributivity in set of orientations under s-accessibility will be proved.

Key Words: Acyclic orientations, Flip-Flop, s-accessibility

Introduction

A graph is a set of [vertices (V), edges (E)] in an ordered pair, where V is nonempty and E is a set of pairs of elements of V. The number of vertices of a graph called its order. If the elements of E are unordered pairs it is called undirected graph, where as one with ordered pairs is known as directed graph. A digraph that has no directed circuit is called acyclic. In this paper directed graphs D = (V, A) with vertex set V and arc set $A \subseteq V \times V$ are taken ,here the loops, i.e. arc of the form (v, v) are forbidden. Also we shall specify vertex and arc set of D as V (D) and A (D) respectively whenever necessary. An undirected graph G = (V, E) with an orientations of G can be directed one i.e. D = (V, A) if there is a bijection f: $E \rightarrow A$ such that f ({u, v}) $\in$ {(u, v), (v, u)}. If D is an orientation of G then G is unique up to isomorphism and we call the isomorphism class of G. An orientation of a finite, undirected, simple, connected graph G with n > 1 vertices assigns a direction to each of its edges so that one of its ends becomes the head of the edge and the other its tail. A vertex is maximal if it is the head of every edge incident with it. Now we define cut as an arc set A(X, V\X) induced by a 2-partition (X, V\X) of the vertex set. The cut consisting of all the arcs that are incident to X and V\X.A cut is called vertex cut if X consists of a single vertex .The set of all cuts of a digraph seen as sign vectors, is integrally spanned by the vertex cuts. The set of non-empty inclusion minimal cuts is denoted by C^*.A cut is positively directed if all its arcs points from X to V\X and it is negatively directed if from V\X to X. Vertices that induce positively and negatively directed vertex cuts are called sources and sinks respectively. We introduce two operations on directed graphs to obtain a partial order on some of the reorientations of a given connected digraph D. Reversing the orientation on all the arcs of a positively directed vertex cut is called a flip, the inverse operation, i.e. reversing the orientation on a negatively directed vertex cut, is called a flop. Felsner and prop studied on partial order on orientations and reorientations of directed structures. Felsner constructs a distributive lattice on those orientations of a planar graph that have fixed out degree on every vertex, while prop present a method.

After a preliminary section on notation and basic results we will describe non-negative integer vector $p\Delta_s(R)$, also we will prove that s-accessibility is anti-symmetric as well as transitive. At last we show that class of orientations forms distributive lattice under the s-accessibility.

Preliminaries

Throughout the paper we fix G as simple connected graph with n > 1 vertices, and R as an acyclic orientation that is orientations for which there are no directed cycles. An orientation S is accessible from R if it is possible to get from R to S by flip-flop sequences of vertex cut $v_1, v_2, v_3, \ldots\ldots . v_k$ the sequence called flip-flop sequence. We fix a vertex v in D and applying flip-flop operation then that vertex is called forbidden vertex. Every reorientation of an orientation can be obtained by flip-flop sequence. To explain partial order set we need to introduce some more terminology. We call a sequence $(v_1, a_1, v_2, a_2, v_3, a_3 \ldots\ldots\ldots\ldots, a_{k-1}, v_k)$ of vertices $v_i \in V$ (D) and arcs $a_i \in A(D)$ a walk if $a_i \in \{(v_i, v_{i+1}), (v_{i+1}, v_i)\}$ for $1 \leq i \leq k-1$ and every arc appears at most once in W . An arc a_i in W is a forward arc if $a_i = (v_i, v_{i+1})$ and backward arc if $a_i = (v_{i+1}, v_i)$. For a walk W denote by F (W) set of its forward arcs. If a walk contains every vertex at most once it is called path or (v_1, v_k)-path. If $v_1 = v_k$ and every other vertex appears at most once in the walk then we call it a cycle. A walk is called directed if it contains no forward or no backward arcs. If a flip-flop sequence of vertex cut does not contain forbidden vertex cut the it is an s-sequence.

Proposition: 1 (Pretzel). If S is accessible from R then R is accessible from S.

Proposition: 2 (Pretzel). If S is accessible from R and they both have the same unique maximal vertex s, then S = R.

Theorem 1

(a) If v occurs in a flip-flop sequence $< v_i >$ from R to S twice, $v = v_i = v_j$ with $i \neq j$ and v' is neighbor of v then $v' = v_k$ for some $i < k < j$.

(b) If the sequence $< v_i >$ is an s-sequence and the distance $\Delta(v, s) = \Delta$, then v occurs in Sequence at most Δ times.

(c) s-accessibility is anti-symmetric.

Proof: (a) When we flip the vertex cut $v = v_i$ then v is replaced by v', in order to apply reverse operation i.e. flop we get again vertex cut v.

(b) We will prove this by induction method if we start with fact that s is prohibited from occurring in the -sequence. Suppose vertices appears $\Delta - 1$ times in sequence and let $v = w_0, w_1, \ldots\ldots, w_\Delta = s$ be a path of minimal length from v to s i.e $\Delta(w, s) = \Delta - 1$ so by induction hypothesis w_i occurs in $v_i, \ldots\ldots\ldots\ldots, v_k$ at most $\Delta - 1$ times .But by the first result w must occur in $v_i, \ldots\ldots\ldots\ldots, v_k$ between any two occurrences of v hence v cannot occur more then Δ times.

(c) This will prove by contradiction method for this late $S \neq R$ i.e. S is not accessible from R, since mutually s-accessible orientations it would be possible to go from R to S and back again arbitrarily often. That would produce s-sequences in which the first vertex v_1 occurred arbitrarily often, contradicting part (b).

Since s-accessibility is obviously transitive, part (c) of the proposition shows that the orientations of our given class form a partially ordered set under $\preccurlyeq_s$, which we shall denote by $(\wp; \preccurlyeq_s)$.

Proposition: 3 There is a unique minimal orientation $\hat{s} \in ((\wp, \preccurlyeq_s)$. It is the only orientation in $\wp$ with s as its unique maximal vertex.

Proof: Let R be any orientation in $\wp$ and suppose that some vertex $v \neq s$ is maximal in R. Then v can be flip-flop and thus R is not minimal. We now introduce the main tool used in our investigation of the partially ordered set $(\wp; \preccurlyeq_s)$. List the vertices of G as $x_1, x_2, \ldots\ldots.. x_n$ To any orientation $R \in \wp$ we define the s-vector $p\Delta_s(R) = (p\Delta_s(R; x_1), p\Delta_s(R; x_2), \ldots\ldots\ldots\ldots\ldots p\Delta_s(R; x_n))$ by latting $p\Delta_s(R; x_i)$ be the number of times x_i is flip-floped in a flipping sequence going from R to $\hat{s}$.Since we are not permitted to flip-flop s we know that $p\Delta_s(R;s)=0$,but we but we retain it in the vector, because it makes it easier to investigate change of sink later. It is not quite trivial that is well defined $p\Delta_s(R; s)$, but that will be established in the next proposition.

Proposition: 4 Let R be an orientation in $\wp$ and let $v \lhd_R w$ be adjacent vertices. Let $<v_i>$ be a flip-flop sequence of cut vertices from R to S. Then v and w occur the same number of times in $<v_i>$ if $v \lhd_R w$ and otherwise w occurs once more thanv.

Proof: The vertices v and w must occur alternately in the sequence. Each time one of them occurs, the position of vw is reversed. Thus if the edge ends with the same direction as it started they must be flip-flop the same number of times. Since w is the head of vw it must be flip-flop before v can be flip-flop. If the position of the edge is changed then one vertex must occur more often than the other and that vertex can only be w.

Corollary:

(a)The number of times a vertex v occurs in any s-sequence from R to $\hat{s}$ is constant $\leq \Delta(v, s)$.

(b) The orientation R is determined by its s-sequence $p\Delta_s(R)$.

Proof: (a) We prove this by induction on $\Delta(v, s)$. If $\Delta(v, s) = 0$, then $v = s$ and the number in question is 0. Otherwise let w be adjacent to v with $\Delta(w, s) = \Delta(v, s) - 1$. Then by induction hypothesis the number of times w occurs in such a sequence is a constant $c \leq \Delta(w, s)$. By the proposition the number of times v occurs is equal to c if the edge vw has the same direction in R and $\hat{s}$. Otherwise it is equal to c + 1 if v is the head of vw in R and equal to c − 1 if w is its head in R. Thus it is also constant. It is at most $\Delta(v, s)$ by Proposition1 (b).

(b) This now follows directly. R is obtained from $\hat{s}$ by reversing precisely those edges xy for which $p\Delta_s(R; x) \neq p\Delta_s(R; y)$.

The set of s-vectors is given a natural (partial) order by setting $p\Delta_s(S) \leq p\Delta_s(R)$ if $p\Delta_s(S,x_i)$, $\leq p\Delta_s(R,x_i)$ for all i. It is fairly obvious that $S \preccurlyeq_s R$ implies $p\Delta_s(S) \leq p\Delta_s(R)$, but in fact the converse also holds.

Theorem 2

The orientation S is s accessible from R if $p\Delta_s(S) \leq p\Delta_s(R)$.

Proof: Each time a forbidden vertex v is flip-flop the value p $\Delta_s(;v)$ is reduced by one, while all other entries in the s-vector remain unchanged. Hence if S is s-accessible from R we have $p\Delta_s(S) \leq p\Delta_s(R)$.

Main Result

The set of orientations in $\wp$ forms a distributive lattice under s-accessibility.

Proof: Consider ($\wp$; $\preccurlyeq_s$) be a poset under s-accessibility for any $X \in \wp$ s-vector are $p\Delta_s(X)$=($p\Delta_s(X;\ x_1)$, $p\Delta_s(X;\ x_2)$,……………. $p\Delta_s(X;\ x_i)$) where $p\Delta_s(X;\ x_i)$ denotes number of times vector x_i is flipped-floped. Let $X_1 \in p\Delta_s(X)$ then $p\Delta_s(X_1)$= { $p\Delta_s(X_1;\ x_1)$, $p\Delta_s(X_1;\ x_2)$,……………. $p\Delta_s(X_1; x_i)$} i.e. $\{n_1, n_2, n_3 \ldots\ldots.. n_i\}$ similarly for $X_2 \in p\Delta_s(X)$ then $p\Delta_s(X_2)$= { $p\Delta_s(X_2;\ x_1)$, $p\Delta_s(X_2;\ x_2)$,……………. $p\Delta_s(X_2;\ x_i)$} i.e $\{p_1, p_2, p_3 \ldots\ldots.. p_i\}$and $X_3 \in p\Delta_s(X)$ then $p\Delta_s(X_3)$= { $p\Delta_s(X_3;\ x_1)$, $p\Delta_s(X_3;\ x_2)$,……………. $p\Delta_s(X_3; x_i)$} i.e $\{q_1, q_2, q_3 \ldots\ldots.. q_i\}$and so on .If $\mathbb{Q}_n$ forms totally ordered set for 0<1<2……….<n-1 where n is number of vertices in G then the entry $p\Delta(X; v)$ of an s-vector is at most $\Delta(v, s) <$ so the vectors form a subset of ${}_n$ so the set is closed under intersections and unions. Hence our set is indeed a sub lattice of ${}_n$ since partially ordered set (; $\preccurlyeq_s$) is order-isomorphic to ($p\Delta_s($ $), \leq$). That set is a sub lattice of ${}_n$ which is distributive.

Conclusion

In this paper we have used s-accessibility, for this we fixed a vertex v in D and by applying flipping and flopping to other vertex named forbidden vertex then constructed sequence of cut vertex called s-sequence This sequence does not contain forbidden vertex, Moreover it is possible to construct such sequence from orientation R to S then we say S is accessible from R to S. So for this accessibility we have assigned a non negative integer vector $p\Delta_s(R)$ to each orientation and showed that ordering of orientations is the natural product of partial order of their vectors which is forms a sub lattice of the product of n copies of the chain 1, 2,…..n. Where n is number of vertices of G which is a distributive lattice.

References

[1] Demendez, O. P. "Orientations bipolaires", PhD Thesis, Paris (**1994**).

[2] Felsner. S., "Convex drawing of planer graphs and the order dimension of 3-polytopes orders", 18, 19-37, (**2001**).

[3] Felsner, S., "Lattice Structures from Planer Graphs", Elector J. Combin, 11, 15-24, (**2004**).

[4] Felsner, S., Kolja, B., Knauer, "Distributive Lattices Polyhedral and Generalized Flow", Technical University, Berlin (**2002**).

[5] Fukuda, K., Prodon, A., Sakuma, T., "Notes on acyclic orientations and the shelling lemma Theory of Computer Science", 263, 9-16, (**2001**).

[6] Kolja, B., Knauer, "Distributive Lattices on Graph Orientations, Technical University Berlin **(2000).**

[7] Latapy. M., Magnien, c., "Coding distributive lattice with edge firing games" Inform. Process Lett.83 (**2000**) 125-128.

[8] Liu, W.P., Rival,I., "Inversions, Cuts and orientations" Discrete Math.87(**1991**) 163-174.

[9] Pretzel. O., "Orientations and reorientations of graphs: Combinatorics and Ordered Sets",Contemp. Math. 57, 103-125,

[10] Propps, J., "Lattice structure for orientation of graphs", **(1993).**

[11] Pretzel, O., "On reoeientating graphs by pushing down maximal vertices order" 3, 135-153 **(1986).**