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A Review and Study of Coupled and Tripled Fixed Point Theory in Partially Ordered Metric Spaces

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Abstract- the aim of this paper is to present a review study of coupled and tripled fixed point in partially ordered complete metrics spaces. A.A.Harandi introduced a new simple and unified approach to coupled and tripled fixed point theory . Our results are review results of A.A.Harandi [13].

Keywords: Fixed Point, Tripled Fixed Point Metric Spaces

I. INTRODUCTION

The existence of a fixed point for contraction type mapping in partially ordered metric spaces has been considered recently by Ran and Reurings [1], Agrawal [2], Bhaskar and Lakshmikantham [3], Lakshmikantham and Ciric [4], Nieto and Lopez[5,6], and Luong and Thuan [7], In [3,7] the authors proved some coupled fixed point theorems in partially ordered sets and found that these results are useful to investigate a large class of problems and have discuss the existence and uniqueness of a solution for initial value problem, periodic boundary value problem and a nonlinear integral equations. In 2010, Samet and Vetro [8] introduced the notion of coupled fixed point and established some new coupled fixed point theorems in complete metric space. Very recently Borcut and Berinde [9] introduced a new concept of a tripled fixed point for a mapping and obtained some existence and uniqueness theorems for contractive type mappings in partially ordered metric spaces.Further, A.A.Harandi [13], introduced a new simple and unified approach to coupled and tripled fixed point theory. They gave a coupled fixed point result for quasi-contractive types maps in partially ordered metric spaces.

II. PRELIMINARIES

We begin this section by some notations, definitions and theorems required in our subsequent discussions.

Definition 2.1([14]) : Let X be a non-empty set. Then $x \in X$ is called a fixed point of a mapping $F: X \to X$ if $F(x) = x$.

Definition 2.2([6]) : Let $(X, \leq)$ be a partially ordered set and $F: X \to X$. We say that F is monotone non-decreasing if $x, y \in X,\ x \leq y \Rightarrow F(x) \leq F(y)$.

Definition 2.3([6]) : Let $(X, \leq)$ be a partially ordered set and $F: X \to X$. We say that F is monotone non-increasing if $x, y \in X,\ x \leq y \Rightarrow F(x) \geq F(y)$.

Definition 2.4([3]) **:** Let $(X,\leq)$ be a partially ordered set and Let $F: X \times X \to X$ be the mapping F is said to have the mixed monotone property if F (x, y) is non-decreasing in x and is non-increasing y i.e. for any x, y ∈ X

$x_1, x_2 \in X$, $x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$ and $y_1, y_2 \in X$, $y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$.

Theorem 2.5([2]) **:**

Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric ρ in X such that (X,ρ) is a complete metric space. Assume there exists a non-decreasing function $\psi: [0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \psi^n(t) = 0$ for each t >0 and also suppose f : X → X is non-decreasing mapping with

$\rho(fx, fy) \leq \psi(\rho(x, y))$, $\forall x \geq y$. Assume that either

(a) f is continuous or

(b) if a non-decreasing sequence $x_n \to x$, then, $x_n \leq x$ $\forall n \in N$.

if there exist $x_0 \in X$ with $x_0 \leq f x_0$, then f has a fixed point. Further more , if for each x, y ∈ X, there exists z ∈ X which is comparable to x and y then the fixed point of f is unique.

Definition 2.6([3]) **:** Let $(X,\leq)$ be a partially ordered set and $F: X \times X \to X$ be the mapping .Then an element (x, y) ∈ X × X is called a coupled fixed point of the mapping F if , F(x, y) = x, and F(y, x) = y .

Definition 2.7([8])**:** Let $(X,\leq)$ be a partially ordered set and

$F: X \times X \times X \to X$ be a map. Then an element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of F if F(x, y, z) = x, F(y, z, x) = y, F(z, x, y) = z.

III. RESULT

We begin this section by reviewing the paper of Bhashkar and Lakshmikanthan [3] in their paper they gave the following results and remarks :

Theorem 3.1([3])**:**

Let $(X,\leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $F: X \times X \to X$ be a continuous mapping having the mixed monotone property on X . Assume that there exists a $k \in [0,1)$ with $d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)]$ for all $x \geq u$ and $y \leq v$. If there exists x_0, $y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist x, y ∈ X such that $x = F(x,y)$ and $y = F(y,x)$.

Theorem 3.2([3])**:**

Let $(X,\leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space . Assume that X has the following property :-

(i). If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x$ for all n.

(ii). If a non-increasing sequence $\{y_n\} \to y$ then $y \leq y_n$ for all n.

Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist $k \in [0,1)$ with $d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)]$ for all x ≥ u and y ≤ v. If there exists x_0, y_0 ∈ X such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist x, y ∈ X such that x = $F(x,y)$ and y = $F(y,x)$.

Remark: . Above theorems cannot gives guarantee of uniqueness of the fixed point.

Further the author A.A.Harandi [13], improves the above results and gave some unique coupled and tripled fixed point theorems along with using the new technique to made the proof of the above results simpler as follows :

Let (X , ≤) be a partially ordered set . Consider the product space $X \times X$with the following partial order :

for (x, y), (u, v) ∈X ×X, (x, y) ≤(u, v) ⇔x ≤u and y ≤ v.

Theorem 3.3([13]):

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space . Let $F: X \times X \to X$ be the mapping having the mixed monotone property on X such that there is a $k \in [0,1)$ with $d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)]$ for all x ≥u and y ≤v . Assume that either

(a) F is continuous or
(b) If a non-decreasing sequence $(x_n, y_n) \to (x,y)$, then $(x_n, y_n) \leq (x,y)$

∀n ∈N. If there exist x_0, y_0 ∈ X such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist ($\overline{x}$, $\overline{y}$ ∈ X such that $\overline{x}$ = F ($\overline{x}$, $\overline{y}$)) and $\overline{y}$ =F ($\overline{y}$, $\overline{x}$)

(c) If for each $(x,y),(z,t) \in X \times X$ there exist $(u,v) \in X \times X$ that is comparable to (x,y) and (z,t) then the coupled fixed point ($\overline{x}$, $\overline{y}$) are unique and $\overline{x} = \overline{y}$.

Proof: Let M = $X \times X$ and let ρ be the metric on M which is defined by

$\rho\big((x,y),(u,v)\big) \leq [d(x,u) + d(y,v)]$. Then it can be easily seen that (M, ρ) is a complete metric space. Let $T: M \to M$ be defined by

$T(x,y) = \big(F(x,y), F(y,x)\big)$. Since F has the mixed monotone then T is non-decreasing . Also by the theorem we have $\rho((Tu,Tv)) \leq \kappa[\rho(u,v)]$, for each $u, v \in M$ with $u \geq v$. since it is assume that either T is continuous or a non-decreasing sequence $u_n \to u, u_n \in M$. again they have $u_n \leq u$. Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then $(x_0, y_0) \leq T(x_0, y_0)$. Also, for each u , $v \in M$, there exist $w \in M$ which is comparable to u and v . Then Theorem 2.5 shows that T has a unique fixed point $\overline{u}$ = ($\overline{x}, \overline{y}$) . Then ($\overline{x}, \overline{y}$) is the unique coupled fixed point of F . Since ($\overline{x}, \overline{y}$) is a coupled fixed point of F then by the definition we have that ($\overline{y}, \overline{x}$) is also a coupled fixed point. Then by uniqueness , we get ($\overline{x}, \overline{y}$) = ($\overline{y}, \overline{x}$) and so $\overline{x} = \overline{y}$.

By the similar argument as the author did in the proof of the theorem 3.3 following corollaries are established .

Corollary 3.4([13]): Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space . Let $F: X \times X \to X$ be the mapping having the mixed monotone property on X such that there is a $\lambda \in [0,1)$ with

$d\big(F(x,y), F(u,v)\big) + d(F(y,x), F(v,u)) \leq$

$\lambda max[d(x,u)+d(y,v), d(x,F(x,y))+d(y,F(y,x)), d(u,F(u,v))+d(v,F(v,u)), d(x,F(u,v))+d(y,F(v,u)), d(u,F(x,y))+d(v,F(y,x))]$ for all $x \geq u$ and $y \leq v$. Assume that either

(a) F is continuous or

(b) If a non-decreasing sequence $(x_n, y_n) \to (x, y)$, then $(x_n, y_n) \leq (x, y)$ $\forall n \in N$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $\overline{x}, \overline{y} \in X$ such that $\overline{x} = (\overline{x}, \overline{y})$ and $\overline{y} = F(\overline{y}, \overline{x})$.

Corollary 3.5([13]):

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that

(X, d) is a complete metric space . Let $F: X \times X \to X$ be the mapping having the mixed monotone property on X such that there is a $\lambda \in [0,1)$ with $d(F(x,y), F(u,v)) + d(F(y,x), F(v,u)) \leq \lambda[d(x,u) + d(y,v)]$

for all $x \geq u$ and $y \leq v$. Assume that either

(a) F is continuous or
(b) If a non-decreasing sequence $(x_n, y_n) \to (x, y)$, then $(x_n, y_n) \leq (x, y)$ $\forall n \in N$. If there exist $x_0, y_0 \in X$ such that

$x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ then there exist $\overline{x}, \overline{y} \in X$ such that $\overline{x} = F(\overline{x}, \overline{y})$ and $\overline{y} = F(\overline{y}, \overline{x})$.

Further the author A.A.Harandi [13], established a new result for tripled fixed point with the help of there technique as follows :

Let $(X, \leq)$ be a partially ordered set. Then partial order on the product space $X \times X \times X \to X$ can be defined as follows

$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \geq v, z \geq w$.

Theorem 3.7([13]):

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that

(X, d) is a complete metric space . Assume there exist non-decreasing functions $\psi_i: [0, \infty) \to [0, \infty)$, i = 1, 2, 3 such that $\psi = \psi_1 + \psi_2 + \psi_3$ is convex,

$\psi(0) = 0$ and $\lim_{n\to\infty} \psi^n(t) = 0$ for each t>0 . Let $F : X \times X \times X \to X$ be a mapping which is non-decreasing in each of its variables and satisfying

$d(F(x,y,z), F(u,v,w)) \leq [\psi_1(d(x,u)) + \psi_2(d(y,v)) + \psi_3(d(z,w))]$ (a) for each $x \geq u, y \geq v, z \geq w$. Suppose either

(a) F is continuous or
(b) If a non-decreasing sequence $(x_n, y_n, z_n) \to (x, y, z)$, then

$(x_n, y_n, z_n) \leq (x, y, z)$ $\forall n \in N$.

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \leq F(z_0, x_0, y_0)$ and $z_0 \leq F(y_0, z_0, x_0)$ then there exist $\overline{x}, \overline{y}, \overline{z} \in X$ such that $F(\overline{x}, \overline{y}, \overline{z}) = \overline{x}, F(\overline{y}, \overline{z}, \overline{x}) = \overline{y}, F(\overline{z}, \overline{x}, \overline{y}) = \overline{z}$

(c) If for each $(x, y, z), (r, s, t) \in X \times X \times X$ there exist

$(u, v, w) \in X \times X \times X$ that is comparable to (x, y, z) and (r, s, t) then the tripled fixed point $(\overline{x}, \overline{y}, \overline{z})$ of F are unique and $\overline{x} = \overline{y} = \overline{z}$.

Proof: Let $M = X \times X \times X$ and let ρ be the metric on M which is defined by

$\rho\big((x, y, z), (u, v, w)\big) = [d(x, u) + d(y, v) + d(z, w)]$ then it can be easily seen that (M, ρ) is a metric space . Let $T: M \to M$ be defined by

$T(x, y, z) = \big(F(x, y, z), F(y, z, x), F(z, x, y)\big)$. Then from (a) and since ψ is superadditive i.e. $\psi: R_+ \to R_+$ is said to be superadditive if

$\psi(s) + \psi(t) \leq \psi(s + t)$, for all $s, t \in R_+$ implies that

$$\rho(T(x, y, z), T(u, v, w)) = d(F(x, y, z), F(u, v, w)) + d(F(y, z, x), F(v, w, u)) + d(F(z, x, y), F(w, u, v))$$

$$\leq \psi_1\big(d(x, u)\big) + \psi_2\big(d(y, v)\big) + \psi_3\big(d(z, w)\big) + \psi_3\big(d(x, u)\big) + \big(d(y, v)\big) + \psi_2\big(d(z, w)\big) + \psi_2\big(d(x, u)\big) + \psi_3\big(d(y, v)\big) + \psi_1\big(d(z, w)\big)$$

$$= \psi\big(d(x, u)\big) + \psi\big(d(y, v)\big) + \psi\big(d(z, w)\big)$$

$$\leq \psi\big(d(x, u)\big) + \big(d(y, v)\big) + \big(d(z, w)\big)$$

$$= \psi(\rho(x, y, z), (u, v, w)))$$

Since F is non-decreasing in each of its variables then T is non-decreasing. Since it is assume that either T is continuous or if a non-decreasing sequence $u_n \to u, u_n \in M.$ again author have $u_n \leq u$. Also, $(x_0, y_0, z_0) \leq T(x_0,\ y_{0,}, z_0)$. Then all assumption of Theorem 2.5 are satisfied . Thus T has a fixed point $(\overline{x}, \overline{y}, \overline{z})$ and so $(\overline{x}, \overline{y}, \overline{z})$ is a tripled fixed point of F. Now suppose condition (c) holds . Then for each $u, v \in M,$ there exists $w \in M$ which is comparable to x and y . Thus by Theorem 2.5, the fixed point of T is unique and so $(\overline{x}, \overline{y}, \overline{z})$ is the unique tripled fixed point of F. Since $(\overline{y}, \overline{z}, \overline{x})$ and $(\overline{z}, \overline{x}, \overline{y})$ are tripled fixed point of T too then by the uniqueness we get $\overline{x} = \overline{y} = \overline{z}$.

IV. CONCLUSION

In this paper, author gave a simple and straight forward proof of coupled and tripled fixed point theorems. .The distinctive feature of this paper is that there technique can easily deduce a new N-order fixed point results in partially ordered metric space , and study of existence of a unique solution for an initial value problem for tripled fixed points.

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