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Optimal Control for Singular Systems via Hybrid Functions

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***Abstract*—**By using hybrid functions of general block-pulse functions and Legendre polynomials, the linear-quadratic problem of time-varying singular systems are transformed into the optimization problem of multivariate functions. The approximate solutions of the optimal control and state as well as the optimal value of the objective functional are derived. The numerical examples are worked out.

***Keywords*—**General Block-Pulse Functions; Legendre Polynomials; Singular Systems; Optimal Control

I. INTRODUCTION

Singular systems are a class of important systems. It is known that the analytic solutions of the optimal control problem for singular systems are difficult to be obtained. This encourages us to find some numerical approach to solve the problem. There are many orthogonal functions or polynomials, such as block-pulse functions and Walsh functions [1-4], Laguerre polynomials [5], Chebyshev polynomials [6], Legendre polynomials [7], Hermite polynomials [8], Fourier series [9] and Legendre wavelets [10], which were used to derive solutions of some systems. In recent years hybrid functions [11] were applied. In this article we use the hybrid functions consisting of general block-pulse functions and Legendre polynomials to solve time-varying singular systems and optimal control for time-varying singular systems. We present the general operational matrices. The numerical solutions are derived by hybrid functions.

II. PRELIMINARIES

A set of block-pulse function $b_k(t)$, $k=1,2,\dots,K$, on the interval $[t_0,T)$ are defined as

$$b_k(t)=\begin{cases}1, & t_{k-1}\le t\le t_k,\\ 0, & \text{otherwise}\end{cases}$$

(1)

where $t_K=T$ and $[t_{k-1},t_k)\subset[t_0,T)$, $k=1,2,\dots,K$.

The Legendre polynomials $L_m(t)$ on the interval $[-1,1]$ are given by the following recursive formula for $m=1,2,\dots,$

$$\begin{cases}L_0(t)=1, \quad L_1(t)=t,\\ (m+1)L_{m+1}(t)=(2m+1)tL_m(t)-mL_{m-1}(t).\end{cases}$$

(2)

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The hybrid functions $h_{km}(t)$, $k=1,2,\dots,K;\ m=0,1,\dots,M-1$, on the interval $[t_0,T)$ are defined as

$$h_{km}(t)=b_k(t)L_m(d_k^{-1}(2t-t_{k-1}-t_k)).$$

(3)

So

$$h_{km}(t)=\begin{cases}L_m(d_k^{-1}(2t-t_{k-1}-t_k)), & t_{k-1}\le t\le t_k,\\ 0, & \text{otherwise}\end{cases}$$

(4)

where $d_k=t_{k-1}-t_k$, $k=1,2,\dots,K.$

Let

$$H_k(t)=[h_{k0}(t),\dots,h_{k,M-1}(t)]^\tau,$$
$$H(t)=[H_1^\tau(t),\dots,H_K^\tau(t)]^\tau,$$

(5)

where τ is the transpose. The operational property of hybrid functions is described by

$$\int_{t_0}^{t}H(s)ds\approx PH(t)$$

(6)

where

$$P=\frac{1}{2}\operatorname{diag}(d_1,\dots,d_K)\otimes\hat{P}+\sum_{k=1}^{K-1}\sum_{i=1}^{K-k}d_i e_{i,i+k}^{(K)}\otimes e_{11}^{(M)},$$

(7)

$$\hat{P}=e_{11}^{(M)}+\sum_{k=1}^{K-1}\left(\frac{1}{2k-1}e_{k,k+1}^{(M)}-\frac{1}{2k+1}e_{k+1,k}^{(M)}\right),$$

and $e_{ij}^{(m)}$ is the $m\times m$ matrix with 1 at its entry $(i,\ j)$ and zeros elsewhere and $\otimes$ denotes Kronecker product.

An l-dimensional vector function $f(t)$ on the interval $[t_0,T)$ is expressed as

$$f(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} f_{km}h_{km}(t)$$

(8)

where

$$f_{km} = \frac{2m+1}{d_k}\int_{t_{k-1}}^{t_k} f(t)L_m(d_k^{-1}(2t-t_{k-1}-t_k))dt.$$

(9)

Rewrite $f(t)$ as

$$f(t) \approx \sum_{k=1}^{K} F_k H_k(t) = FH(t)$$

(10)

where

$$F_k = [f_{k0},\dots,f_{k,M-1}], \quad F = [F_1,\dots,F_K].$$

(11)

For corresponding F_k and F we denote

$$\hat{F}_k = [f_{k0}^{\tau},\dots,f_{k,M-1}^{\tau}]^{\tau}, \qquad \hat{F} = [\hat{F}_1^{\tau} \;,\dots,\hat{F}_K^{\tau}]^{\tau} \;.$$

(12)

Let a matrix function $M(t)$ be appropriate to a vector function $f(t)$. We express $M(t)$ and $f(t)$, respectively, as

$$M(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} M_{km}h_{km}(t)\,, \; f(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} f_{km}h_{km}(t)\,,$$

(13)

then

$$M(t) \approx \sum_{k=1}^{K}\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} M_{ki} f_{kj} h_{ki}(t) h_{kj}(t).$$

From

$$h_{ki}(t)h_{kj}(t) = \sum_{m=0}^{M-1} d_{km}^{(ij)} h_{km}(t)$$

(14)

where

$$d_{km}^{(ij)} = \frac{2m+1}{2}\int_{-1}^{1} L_i(t)L_j(t)L_m(t)dt,$$

we have

$$M(t)f(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1}\left(\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} d_{km}^{(ij)} M_{ki} f_{kj}\right) h_{km}(t)$$
$$= \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{M}_{km}h_{km}(t) = \sum_{k=1}^{K} \tilde{M}_k H_k(t)$$

(15)

where

$$\tilde{M}_{km} = \sum_{i=0}^{M-1}\sum_{j=0}^{M-1} d_{km}^{(ij)} M_{ki} f_{kj} = \hat{M}_{km} \qquad \hat{F}_k \qquad ,$$

(16)

$$\tilde{M}_k = [\tilde{M}_{k0},\dots,\tilde{M}_{k,M-1}],$$

$$\hat{M}_{km} = \left[\sum_{j=0}^{M-1} d_{km}^{(i0)} M_{ki},\dots,\sum_{j=0}^{M-1} d_{km}^{(i,M-1)} M_{ki}\right].$$

Let

$$\hat{M}_k \quad = [\hat{M}_{k0}^{\tau} \quad ,\dots,\hat{M}_{k,M-1}^{\tau}]^{\tau},$$

(17)

$$\hat{\tilde{M}}_k \quad = [\tilde{M}_{k0}^{\tau} \quad ,\dots,\tilde{M}_{k,M-1}^{\tau}]^{\tau}.$$

(18)

Then

$$\hat{\tilde{M}}_k \qquad = \hat{M}_k \qquad \hat{F}_k \qquad .$$

(19)

III. Analysis of Time-Varying Singular Systems

Consider the following time-varying singular system

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0$$

(20)

where $x(t)$ is the n-dimensional state function and $u(t)$ *r*-dimensional control function. $E(t)$, $A(t)$ and $B(t)$ are the matrix functions of appropriate dimensions. $E(t)$ is singular. Suppose that (20) has the unique solution for a given $u(t)$. We express $E(t)$, $\dot{x}(t)$, $A(t)$, $x(t)$, $B(t)$ and $u(t)$, respectively, as

$$E(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} E_{km}h_{km}(t) \;\;,\;\; \dot{x}(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} x_{km}^{*}h_{km}(t) \;\;,$$

(21)

$$A(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} A_{km}h_{km}(t) \;\;,\;\; x(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} x_{km}h_{km}(t) \;\;,$$

(22)

$$B(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} B_{km}h_{km}(t) \;\;,\;\; u(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} u_{km}h_{km}(t) \;\;.$$

(23)

By (15) we obtain

$$E(t)\dot{x}(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{E}_k H_k(t) \qquad ,$$

(24)

$$A(t)x(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{A}_k H_k(t),$$

$$B(t)u(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{B}_k H_k(t).$$

Substituting (24) into (20) we have

$$[\tilde{E}_1,\dots,\tilde{E}_K]H(t) = \{[\tilde{A}_1,\dots,\tilde{A}_K] + [\tilde{B}_1,\dots,\tilde{B}_K]\}H(t).$$

Therefore

$$[\hat{\tilde{E}}_1^\tau ,..., \hat{\tilde{E}}_K^\tau]^\tau = [\hat{\tilde{A}}_1^\tau ,..., \hat{\tilde{A}}_K^\tau]^\tau + [\hat{\tilde{B}}_1^\tau ,..., \hat{\tilde{B}}_K^\tau]^\tau$$

where

$$\hat{\tilde{E}}_k = [\tilde{E}_{k0}^\tau ,..., \tilde{E}_{k,M-1}^\tau]^\tau,$$

$$\hat{\tilde{A}}_k = [\tilde{A}_{k0}^\tau ,..., \tilde{A}_{k,M-1}^\tau]^\tau,$$

$$\hat{\tilde{B}}_k = [\tilde{B}_{k0}^\tau ,..., \tilde{B}_{k,M-1}^\tau]^\tau.$$

By (17) we have

$$[(\hat{E}_1 \ \hat{X}_1^*)^\tau ,..., (\hat{E}_K \ \hat{X}_K^*)^\tau]^\tau = [(\hat{A}_1 \ \hat{X}_1)^\tau ,..., (\hat{A}_K \ \hat{X}_K)^\tau]^\tau + [(\hat{B}_1 \ \hat{X}_1)^\tau ,..., (\hat{B}_K \ \hat{X}_K)^\tau]^\tau .$$

where $\hat{E}_K$, $\hat{A}_K$ and $\hat{B}_K$ have the similar meaning as (18) and (19). In addition,

$$\hat{X}_k^* = [x_{k0}^{*\ \tau} ,..., x_{k,M-1}^{*\ \tau}]^\tau,$$

$$\hat{X}_k = [x_{k0}^\tau ,..., x_{k,M-1}^\tau]^\tau,$$

$$\hat{U}_k = [u_{k0}^\tau ,..., u_{k,M-1}^\tau]^\tau.$$

Using Kronecker product we rewrite the above equation as

$$\sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{E}_k) \quad \hat{X}^* = \sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{A}_k) \quad \hat{X} + \sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{B}_k) \ \hat{U} \qquad (25)$$

where

$$\hat{X}^* = [\hat{X}_1^{*\tau} ,..., \hat{X}_K^{*\ \tau}]^\tau,$$

$$\hat{X} = [\hat{X}_1^\tau ,..., \hat{X}_K^\tau]^\tau,$$

$$\hat{U} = [\hat{U}_1^\tau ,..., \hat{U}_K^\tau]^\tau.$$

By (6) we have

$$XH(t) - X_0 H(t) \approx X^* PH(t).$$

Using Kronecker product we have

$$(P^\tau \otimes I_n)\hat{X}^* = \hat{X} - \hat{X}_0$$

where

$$\hat{X}_0 = [\hat{X}_{01}^\tau ,..., \hat{X}_{0K}^\tau]^\tau,$$

$$\hat{X}_{0k} = [x^\tau(t_0), 0^\tau, ..., 0^\tau]^\tau, \ k = 1, 2, ..., K.$$

So

$$\sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{E}_k) \ \hat{X}^* = \Delta \ (\hat{X} - \hat{X}_0)$$

where

$$\Delta = \sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{E}_k) \ [(P^{-1})^\tau \otimes I_n]$$

Applying singular value decomposition to Δ, we have

$$S\Delta V^\tau = \begin{bmatrix} \Omega & 0 \\ 0 & 0_{(MKn-N)\times(MKn-N)} \end{bmatrix} \qquad (26)$$

where

$$\Omega = \mathrm{diag}(\sigma_1, ..., \sigma_N) \ , \ \sigma_k \neq 0 \ , \quad k = 1, ..., N. \qquad (27)$$

Let

$$\hat{X} - \hat{X}_0 = V^\tau \hat{Z} = V^\tau [\hat{Z}_1^\tau, \hat{Z}_2^\tau]^\tau,$$

$$S\sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{A}_k) \ \hat{X}_0 = [\hat{X}_{01}^\tau, \hat{X}_{02}^\tau]^\tau,$$

$$(V^\tau)^{-1} = \begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{bmatrix}, \ S\sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{B}_k) = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \qquad (28)$$

$$S\sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{A}_k) \quad V^\tau = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \qquad (29)$$

where $\hat{Z}$ is the *MKn*-dimensional vector; $\hat{Z}_1$ and $\hat{X}_{01}$ are *N*-dimensional vectors. $\tilde{V}_{11}$ and W_{11} are $N \times N$ matrices and L_1 an $N \times Kr$ matrix. Combining (25) and (28), we have

$$S\Delta V^\tau \hat{Z} = S\sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{A}_k) \quad V^\tau \hat{Z} + S\sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{B}_k) \ \hat{U} + S\sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{A}_k) \ V^\tau \hat{X}_0 . \qquad (30)$$

Rewrite (30) as

$$\begin{cases} \Omega \hat{Z}_1 = W_{11} \hat{Z}_1 + W_{12} \hat{Z}_2 + \hat{X}_{01}, \\ W_{21} \hat{Z}_1 + W_{22} \hat{Z}_2 + L_2 \hat{U} + \hat{X}_{02} . \end{cases} \qquad (31)$$

Let

$$\hat{Y} = \Omega \hat{Z}_1 \ . \qquad (32)$$

Then

$$\hat{Y} = C\hat{Y} + D\hat{U} + \hat{Y}_0 \qquad (33)$$

where

$$\begin{cases} C = (W_{11} - W_{12}W_{22}^{-1}W_{21})\Omega^{-1}, \\ D = L_1 - W_{12}W_{22}^{-1}L_2, \\ \hat{Y}_0 = \hat{X}_{01} - W_{12}W_{22}^{-1}\hat{X}_{02}. \end{cases}$$

(34)

By (28) and (31) we have

$$\hat{X} = V^{\tau}\begin{bmatrix} \hat{Z}_1 \\ \hat{Z}_2 \end{bmatrix} + \hat{X}_0$$

$$= V^{\tau}\begin{bmatrix} \Omega^{-1}\hat{Y} \\ W_{22}^{-1}[-W_{21}\Omega^{-1}\hat{Y} - L_2\hat{U} - \hat{X}_{02}] \end{bmatrix} + \hat{X}_0$$

$$+V^{\tau}\begin{bmatrix} I_N \\ -W_{22}^{-1}W_{21} \end{bmatrix} + \Omega^{-1}\hat{Y}$$

$$+V^{\tau}\begin{bmatrix} 0 \\ -W_{22}^{-1}L_2 \end{bmatrix}\hat{U}$$

$$+V^{\tau}\begin{bmatrix} 0 \\ -W_{22}^{-1}\hat{X}_{02} \end{bmatrix} + \hat{X}_0.$$

Substituting (33) into the above equation we have

$$\hat{X} = \Phi\hat{U} + G$$

(35)

where

$$\Phi = V^{\tau}[I_N, -(W_{22}^{-1}W_{21})^{\tau}]^{\tau}\Omega^{-1}(I_N - C)^{-1}D$$
$$+V^{\tau}[0^{\tau}, -(W_{22}^{-1}L_2)^{\tau}]^{\tau} ,$$

(36)

$$G = V^{\tau}[I_N, -(W_{22}^{-1}W_{21})^{\tau}]^{\tau}\Omega^{-1}(I_N - C)^{-1}\hat{Y}_0$$
$$+V^{\tau}[0^{\tau}, -(W_{22}^{-1}\hat{X}_{02})^{\tau}]^{\tau} + \hat{X}_0 .$$

(37)

IV. Optimal control problem

Given a time-varying singular system described by (20), we want to find the optimal control which minimizes the cost functional

$$J = \frac{1}{2}\int_{t_0}^{T}[x^{\tau}(t)Q(t)x(t) + u^{\tau}(t)R(t)u(t)]dt$$

(38)

where the matrices $Q(t)$ and $R(t)$ are positive-semidefinite and positive-definite, respectively, for any $t \in [t_0, T]$. We express $x(t)$, $u(t)$, $Q(t)x(t)$ and $R(t)u(t)$, respectively, as

$$x(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} x_{km}h_{km}(t) ,\ u(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} u_{km}h_{km}(t) ,$$

$$Q(t)x(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{Q}_{km}h_{km}(t) ,$$

$$R(t)u(t) \approx \sum_{k=1}^{K}\sum_{m=0}^{M-1} \tilde{R}_{km}h_{km}(t) .$$

Substituting the above equations into (38) we have

$$J = \frac{1}{2}\sum_{k=1}^{K}\sum_{m=0}^{M-1}(x_{km}^{\tau}\tilde{Q}_{km} + u_{km}^{\tau}\tilde{R}_{km})\int_{t_{k-1}}^{t_k} h_{km}^2(t)dt .$$

Rewrite the above equation as

$$J = \frac{1}{2}(\hat{X}^{\tau} \quad \bar{Q} \quad \hat{X} + \quad \hat{U}^{\tau} \quad \bar{R} \quad \hat{U})$$

(39)

where

$$\bar{Q} = \sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{Q}_k) ,$$

(40)

$$\hat{Q}_k = \quad d_k[\hat{Q}_{k0}^{\tau}, \ \frac{1}{3}\hat{Q}_{k1}^{\tau}, ..., \ \frac{1}{2M+1}\hat{Q}_{k,M-1}^{\tau}] ,$$

(41)

$$\hat{Q}_{km} = \left[\sum_{j=0}^{M-1} d_{km}^{(i0)}Q_{ki}, ..., \sum_{j=0}^{M-1} d_{km}^{(i,M-1)}Q_{ki}\right] ,$$

(42)

$$\bar{R} = \sum_{k=1}^{K}(e_{kk}^{(K)} \otimes \hat{R}_k) ,$$

(43)

$$\hat{R}_k = \quad d_k[\hat{R}_{k0}^{\tau}, \ \frac{1}{3}\hat{R}_{k1}^{\tau}, ..., \ \frac{1}{2M+1}\hat{R}_{k,M-1}^{\tau}] ,$$

(44)

$$\hat{R}_{km} = \left[\sum_{j=0}^{M-1} d_{km}^{(i0)}R_{ki}, ..., \sum_{j=0}^{M-1} d_{km}^{(i,M-1)}R_{ki}\right] .$$

(45)

By means of $\frac{\partial J}{\partial \hat{U}} = 0$, we have $\Phi^{\tau}\bar{Q}\hat{X} + \ \bar{R}\hat{U} = 0$. Then

$$\hat{U} = \quad -\bar{R}^{-1}\Phi^{\tau}\bar{Q}\hat{X} .$$

(46)

From (35) we have

$$\hat{X} = (I_{MKn} + \Phi\bar{R}^{-1}\Phi^{\tau}\bar{Q})^{-1}G ,$$

(47)

$$\hat{U} = \quad -\bar{R}^{-1}\Phi^{\tau}\bar{Q} \quad (I_{MKn} + \Phi\bar{R}^{-1}\Phi^{\tau}\bar{Q})^{-1}G .$$

(48)

V. NUMERICAL EXAMPLES

Example 1: Consider the time-varying singular system described by

$$\begin{bmatrix} t & 0 \\ t & 0 \end{bmatrix}\dot{x}(t)=\begin{bmatrix} -2t^2-2 & 2t^2 \\ 0 & 2 \end{bmatrix}x(t)+\begin{bmatrix} 2 \\ 2 \end{bmatrix}u(t),$$

$$x(0)=[-1,1]^\tau.$$

Let $u(t)=-1$ and $T=1$. Then by (35) and taking $K=3$, $t_k=k/K$, $k=0,1,\ldots,K$ and $M=4$ we have the hybrid solution $x(t)=[x_1(t),\, x_2(t)]^\tau$:

$$\begin{aligned} x_1(t) &= -\frac{26}{27}h_{10}(t)+\frac{1}{18}h_{11}(t)+\frac{1}{54}h_{12}(t) \\ &-\frac{20}{27}h_{20}(t)+\frac{1}{6}h_{21}(t)+\frac{1}{54}h_{22}(t) \\ &-\frac{8}{27}h_{30}(t)+\frac{5}{18}h_{31}(t)+\frac{1}{54}h_{32}(t), \\ x_2(t) &= \frac{28}{27}h_{10}(t)+\frac{1}{18}h_{11}(t)+\frac{1}{54}h_{12}(t) \\ &+\frac{34}{27}h_{20}(t)+\frac{1}{6}h_{21}(t)+\frac{1}{54}h_{22}(t) \\ &+\frac{46}{27}h_{30}(t)+\frac{5}{18}h_{31}(t)+\frac{1}{54}h_{32}(t). \end{aligned}$$

It can be checked that the hybrid solutions are equal to the exact solutions.

Example 2: Consider the time-varying singular system described by

$$\begin{bmatrix} 2t^2+2 & t^2+1 \\ 0 & 0 \end{bmatrix}\dot{x}(t)=\begin{bmatrix} t^2 & t+1 \\ 0 & 1 \end{bmatrix}x(t)+\begin{bmatrix} 1 \\ t^2+1 \end{bmatrix}u(t),$$

$$x(0)=[1,1]^\tau.$$

where $x(t)$ is 2-dimensional state function and $u(t)$ 1-dimensional control function, we want to find the optimal control which minimizes the cost functional

$$J=\frac{1}{2}\int_0^1[x^\tau(t)\begin{bmatrix} 1 & t \\ t & t^2 \end{bmatrix}x(t)+(t^2+1)u^2(t)]dt.$$

By (47) and (48), taking $K=3$, $t_k=k/K$, $k=0,1,\ldots,K$ and $M=5$ we have the hybrid solution $x(t)=[x_1(t),\, x_2(t)]^\tau$ and $u(t)$:

$$\begin{aligned} x_1(t) &= \frac{2065}{1557}h_{10}(t)+\frac{130}{963}h_{11}(t)+\frac{55}{6602}h_{12}(t) \\ &-\frac{28}{67489}h_{13}(t)-\frac{3}{230560}h_{14}(t)+\frac{941}{575}h_{20}(t) \\ &+\frac{1483}{8651}h_{21}(t)+\frac{56}{16523}h_{22}(t)-\frac{21}{27253}h_{23}(t) \\ &-\frac{14}{144535}h_{24}(t)+\frac{5998}{3039}h_{30}(t)+\frac{328}{2123}h_{31}(t) \\ &-\frac{168}{16477}h_{32}(t)-\frac{31}{22855}h_{33}(t)+\frac{29}{192550}h_{34}(t) \end{aligned}$$

,

$$\begin{aligned} x_2(t) &= \frac{829}{2310}h_{10}(t)-\frac{941}{3719}h_{11}(t)-\frac{37}{3269}h_{12}(t) \\ &+\frac{21}{38632}h_{13}(t)-\frac{4}{90193}h_{14}(t)-\frac{263}{1279}h_{20}(t) \\ &-\frac{1733}{5629}h_{21}(t)-\frac{34}{5401}h_{22}(t)+\frac{19}{16771}h_{23}(t) \\ &+\frac{1}{4099}h_{24}(t)-\frac{635}{784}h_{30}(t)-\frac{681}{2492}h_{31}(t) \\ &+\frac{435}{19963}h_{32}(t)+\frac{25}{7506}h_{33}(t)-\frac{23}{94536}h_{34}(t) \end{aligned}$$

,

$$\begin{aligned} u(t) &= -\frac{638}{1819}h_{10}(t)+\frac{212}{813}h_{11}(t)+\frac{99}{12532}h_{12}(t) \\ &-\frac{86}{24177}h_{13}(t)+\frac{15}{190621}h_{14}(t)+\frac{221}{1440}h_{20}(t) \\ &+\frac{211}{943}h_{21}(t)-\frac{89}{5284}h_{22}(t)-\frac{53}{34566}h_{23}(t) \\ &+\frac{16}{323711}h_{24}(t)+\frac{1631}{3464}h_{30}(t)+\frac{439}{5163}h_{31}(t) \\ &-\frac{198}{7313}h_{32}(t)+\frac{18}{152083}h_{33}(t)+\frac{22}{77923}h_{34}(t) \end{aligned}$$

.

By (39) we have $J=\dfrac{624}{581}$.

Example 3: Consider the time-varying singular system described by

$$\begin{bmatrix} 1 & t^2+1 & t^2+1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\dot{x}(t)=\begin{bmatrix} t & 1 & t^2+1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}x(t)$$

$$+\begin{bmatrix}1 & t+1\\ t & 1\\ 1 & t^2+1\end{bmatrix}u(t),\ x(0)=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

where $x(t)$ is 3-dimensional state function and $u(t)$ 2-dimensional control function, we want to find the optimal control which minimizes the cost functional

$$J=\frac{1}{2}\int_0^1[x^\tau(t)\begin{bmatrix}t^2 & 0 & 0\\ 0 & t^2 & t\\ 0 & t & 1\end{bmatrix}x(t)+u^\tau(t)u(t)]dt.$$

Taking $M=5$, $K=3$, $t_k=k/K$, $k=0,1,...,K$, and by (47) and (48) we have the hybrid solutions $x(t)=[x_1(t),x_2(t),x_3(t)]^\tau$ and $u(t)=[u_1(t),u_2(t)]^\tau$:

$$x_1(t)=\frac{829}{967}h_{10}(t)-\frac{818}{5131}h_{11}(t)-\frac{82}{8589}h_{12}(t)+\frac{62}{8647}h_{13}(t)-\frac{3}{28075}h_{14}(t)+\frac{498}{829}h_{20}(t)-\frac{363}{5491}h_{21}(t)+\frac{133}{4567}h_{22}(t)+\frac{9}{28057}h_{23}(t)-\frac{6}{18877}h_{24}(t)+\frac{194}{307}h_{30}(t)+\frac{228}{2551}h_{31}(t)+\frac{475}{24611}h_{32}(t)-\frac{80}{37167}h_{33}(t)-\frac{7}{29461}h_{34}(t),$$

$$x_2(t)=\frac{1335}{962}h_{10}(t)+\frac{1003}{1714}h_{11}(t)-\frac{165}{12643}h_{12}(t)-\frac{170}{8397}h_{13}(t)+\frac{32}{15715}h_{14}(t)+\frac{1251}{544}h_{20}(t)+\frac{929}{2881}h_{21}(t)-\frac{284}{11065}h_{22}(t)+\frac{332}{41737}h_{23}(t)-\frac{18}{26195}h_{24}(t)+\frac{431}{150}h_{30}(t)+\frac{502}{1939}h_{31}(t)-\frac{87}{5035}h_{32}(t)-\frac{611}{92363}h_{33}(t)-\frac{29}{26141}h_{34}(t),$$

$$x_3(t)=\frac{203}{250}h_{10}(t)-\frac{181}{511}h_{11}(t)+\frac{215}{5003}h_{12}(t)+\frac{559}{30217}h_{13}(t)-\frac{41}{16978}h_{14}(t)+\frac{452}{919}h_{20}(t)+\frac{56}{2677}h_{21}(t)+\frac{127}{3553}h_{22}(t)-\frac{129}{14219}h_{23}(t)+\frac{18}{21409}h_{24}(t)+\frac{1618}{2515}h_{30}(t)+\frac{656}{5547}h_{31}(t)+\frac{277}{13523}h_{32}(t)+\frac{83}{13044}h_{33}(t)+\frac{13}{10950}h_{34}(t),$$

$$u_1(t)=-\frac{4000}{4459}h_{10}(t)+\frac{1134}{3049}h_{11}(t)+\frac{999}{16883}h_{12}(t)-\frac{205}{8772}h_{13}(t)-\frac{157}{81143}h_{14}(t)-\frac{383}{1718}h_{20}(t)+\frac{827}{4136}h_{21}(t)-\frac{1395}{21623}h_{22}(t)+\frac{97}{20900}h_{23}(t)+\frac{120}{90703}h_{24}(t)-\frac{364}{6751}h_{30}(t)+\frac{245}{7569}h_{31}(t)+\frac{254}{19571}h_{32}(t)+\frac{45}{5293}h_{33}(t)+\frac{11}{182203}h_{34}(t),$$

$$u_2(t)=-\frac{2137}{2680}h_{10}(t)-\frac{405}{2821}h_{11}(t)+\frac{228}{1831}h_{12}(t)-\frac{125}{61771}h_{13}(t)-\frac{29}{11068}h_{14}(t)-\frac{315}{583}h_{20}(t)+\frac{911}{2983}h_{21}(t)+\frac{79}{6051}h_{22}(t)-\frac{47}{4344}h_{23}(t)+\frac{32}{39001}h_{24}(t)-\frac{165}{3628}h_{30}(t)+\frac{753}{6050}h_{31}(t)-\frac{137}{1952}h_{32}(t)-\frac{111}{13087}h_{33}(t)-\frac{13}{42821}h_{34}(t).$$

By (39) we have $J=\frac{1747}{661}$.

VI. CONCLUSION

By the excellent properties of operational matrices of the hybrid function of general block-pulse functions and Legendre polynomials, the general algorithms for the time-varying singular system and the optimal control problem are convenient for application. The examples illustrate this approach.

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