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Error Analysis between Runge Kutta Fourth Order Method & Fehlberg 6th Order Method of Hybrid Fuzzy Fractional Differential Equations

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Abstract— In this paper we study Error analysis of numerical methods for second order hybrid fuzzy fractional differential equations. We solve the hybrid fuzzy fractional differential equations with a fuzzy initial condition by using variational iteration method. We consider a second order differential equation with fractional values and we compared the results with their exact solutions in order to demonstrate the validity and applicability of the method. We further give the definition of the Degree of Sub element hood of hybrid fuzzy fractional differential equations with examples.

Keywords— Hybrid fuzzy fractional differential equations, Degree of Sub Element hood, Runge Kutta Fehlberg 6th order Method

I. INTRODUCTION

With the rapid development of linear and nonlinear science, many different methods such as the variational iteration method (VIM) [1] were proposed to solve fuzzy differential equations. Fuzzy initial value problems for fractional differential equations have been considered by some authors recently [2, 3]. To study some dynamical processes, it is necessary to take into account imprecision, randomness or uncertainty. The uncertainty can be modelled by incorporating it into the dynamical system and considering fuzzy differential equations. The origins of fractional calculus go back to 1695 when Leibniz considered the derivative of order 1/2. In particular, fractional differential equations have received much attention and a number of recent works concern their numerical solution. As another development, hybrid systems are dynamical systems that progress continuously in time but have formatting changes called modes at a sequence of discrete times. Some recent papers about hybrid systems include [6]. When the continuous time dynamics of a hybrid system comes from fuzzy fractional differential equations the system is called a hybrid fuzzy fractional differential system or a hybrid fuzzy fractional differential equation. This is one of the first papers to study hybrid fractional differential equations. The aim of this paper is to study their numerical solution.

This paper is organized as follows. In Section 2, we provide some background on fuzzy fractional differential equations and hybrid fuzzy fractional differential equations. In Section 3 we discuss the numerical solution of Second order hybrid fuzzy fractional differential equations by Runge Kutta 4th order & Runge Kutta 6th order Fehlberg method. The method given uses piecewise application of a numerical method for fuzzy fractional differential equations. In Section 4, as an example, we numerically analyzed the error between the methods for Second order hybrid fuzzy fractional differential equations. The objective of the present paper is to extend the application of the variational iteration method, to provide approximate solutions for fuzzy initial value problems of differential equations of fractional order, and to make comparison with that obtained by an exact fuzzy solution.

II. HYBRID FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS

Preliminaries

In this section the most basic notations used in fuzzy calculus are introduced. We start with defining a fuzzy number.
We now recall some definitions needed through the paper. The basic definition of fuzzy numbers is given by R, we denote the set of all real numbers. A fuzzy number is a mapping $u : R \to [0; 1]$ with the following properties:
(a) u is upper semi-continuous,

(b) u is fuzzy convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq min\{u(x); u(y)\}$ for all $x; y \in R; \lambda \in [0; 1]$,

(c) u is normal, i.e., $\exists x_0 \in R$ for which $u(x_0) = 1$,

(d) supp $u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact. Let E be the set of all fuzzy number on R. The r-level set of a fuzzy number

$u \in E$, $0 \le r \le 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in R \mid u(x) \ge r\} & \text{if } 0 < r \le 1 \\ cl(supp\, u) & \text{if } r = 0 \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r); \overline{u}(r)]$,

where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$ and $\overline{u}(r)$ denotes the right-hand endpoint
of $[u]_r$. Since each $y \in R$ can be regarded as a fuzzy number Y defined by

$$Y(t) = \begin{cases} 1 & \text{if } t = y \\ 0 & \text{if } t \ne y \end{cases}$$

Definition 1.
A fuzzy number (or an interval) u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions

$\underline{u}(r), \overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements :

1. $\underline{u}(r)$ is a bounded non-decreasing left continuous function in (0, 1] and right
continuous at 0.
2. $\overline{u}(r)$ is a bounded non-decreasing left continuous function in (0, 1] and right
continuous at 0.
3. $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

Let us consider the following fractional differential equation:

$${}_cD_a^{\beta} x(t) = f(t, x(t), \lambda_k(x_k)),\ t \in [t_k, t_{k+1}] \quad (1)$$

Where, $0 \le t_0 \le t_1 \le \ldots \le t_k \to \infty$

$$f \in C[R^+ \times E \times E, E], \lambda_k \in C[E, E]$$

Here we assume that the existence and uniqueness of solution of the hybrid system hold on each $[t_k, t_{k+1}]$ to be specific the system would look like:

$${}_cD_a^{\beta} x(t)$$

$$\begin{cases} {}_cD_a^{\beta} x_0(t) = f(t, x_0(t), \lambda_0(x_0)), x(t_0) = x_0, t \in [t_0, t_1] \\ {}_cD_a^{\beta} x_1(t) = f(t, x_1(t), \lambda_1(x_1)), x(t_1) = x_1, t \in [t_1, t_2] \\ \quad . \\ \quad . \\ {}_cD_a^{\beta} x_k(t) = \\ f(t, x_k(t), \lambda_k(x_k)), x(t_k) = x_k, t \in [t_k, t_{k+1}] \end{cases}$$

By the solution of (1) we mean the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), t \in [t_0, t_1] \\ x_1(t), t \in [t_1, t_2] \\ \quad . \\ \quad . \\ x_k(t), t \in [t_k, t_{k+1}] \\ \quad . \\ \quad . \end{cases}$$

We note that the solutions of (1) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in$ E and k = 0, 1, 2…..
We can also represent a fuzzy numbers x ∈ E by a pair of functions

$${}_cD_a^{\beta} x(t) = {}_cD_a^{\beta}\left[\underline{x}(t;r), \overline{x}(t;r)\right]$$
$$= [\ {}_cD_a^{\beta} \underline{x}(t), {}_cD_a^{\beta} \overline{x}(t)\ ]$$

Using a representation of fuzzy numbers we may represent x ϵ E by a pair of functions $\left(\underline{x}(r), \overline{x}(r)\right)$, $0 \le r \le 1$ such that:
1. $\underline{x}(r)$ is bounded, left continuous and non decreasing,
2. $\bar{x}(r)$ is bounded, left continuous and non increasing and
3. $\underline{x}(r) \le \bar{x}(r)$, $0 \le r \le 1$

Therefore, we may replace (1) by an equivalent system equation (2):

$$\begin{cases} {}_cD_a^{\beta} \underline{x}(t) = \underline{f}(t, x, \lambda_k(x_k)) \equiv F_k(t, \underline{x}, \overline{x}), \underline{x}(t_k) = \underline{x_k} \\ {}_cD_a^{\beta} \overline{x}(t) = \overline{f}(t, x, \lambda_k(x_k)) \equiv G_k(t, \underline{x}, \overline{x}), \overline{x}(t_k) = \overline{x_k} \end{cases} \quad \ldots\ldots (2)$$

This possesses a unique solution $\left(\underline{x}, \overline{x}\right)$, *which is a fuzzy function. That is for each t, the pair* $\left[\underline{x}(t;r), \overline{x}(t;r)\right]$ *is a fuzzy number, where* $\left[\underline{x}(t;r), \overline{x}(t;r)\right]$ *are respectively the solutions of the parametric form given by Equation (3):*

$$\begin{cases} {}_cD_a^{\beta} \underline{x}(t) = F_k(t, \underline{x}(t;r), \overline{x}(t;r)), \underline{x}(t_k;r) = \underline{x_k}(r) \\ {}_cD_a^{\beta} \overline{x}(t) = G_k(t, \underline{x}(t;r), \overline{x}(t;r)), \overline{x}(t_k;r) = \overline{x_k}(r) \end{cases} \quad \ldots\ldots (3)$$

for r [0, 1]

III. THE FOURTH ORDER RUNGE KUTTA METHOD WITH HARMONIC MEAN FOR SECOND ORDER DIFFERENTIAL EQUATIONS

For a second order hybrid fuzzy fractional differential equation we develop the fourth order Runge Kutta method with harmonic mean when f and λ_k in (1) can be obtained via the Zadeh extension principle from:

$f \epsilon [R^+ \text{ X } R \text{ X } R, R]$ and $\lambda_k \epsilon C [R,R]$

we assume that the existence and uniqueness of solutions of (1) hold for each $[t_k, t_{k+1}]$. For a fixed r, to integrate the system in (3) $[t_0,t_1],[t_1,t_2],\ldots..[t_k,t_{k+1}]\ldots\ldots$we replace each interval by a set of N_{k+1} discrete equally spaced grid points (including the end points) at which the exact solution

x(t; r)=($\underline{x}(t;r),\overline{x}(t;r)$) is approximated by some ($\underline{y}(t;r),\overline{y}(t;r)$) &($\underline{z}(t;r),\overline{z}(t;r)$) .

For the chosen grid points on $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k$, $h_k = \dfrac{t_{k+1}-t_k}{N_k}$, $0 \le n \le N_k$.

Let $(\underline{Y}_k(t;r),\overline{Y_k}(t;r)) \equiv (\underline{x}_k(t;r),\overline{x_k}(t;r))$, $(\underline{y}_k(t;r),\overline{y_k}(t;r),\underline{z}(t;r),\overline{z}(t;r))$ and ($\underline{y}_k(t;r),\overline{y_k}(t;r)$) may be denoted respectively by ($\underline{Y}_{k,n}(t;r),\overline{Y}_{k,n}(t;r)$) and ($\underline{y}_{k,n}(t;r),\overline{y}_{k,n}(t;r)$).

We allow N_k's to vary over the $[t_k, t_{k+1}]$'s so that the h_k's may be comparable.

The Fourth Order Runge Kutta method for (1) is given by:

$(\underline{Y}_k(t;r),\overline{Y_k}(t;r)) \equiv (\underline{x}_k(t;r),\overline{x_k}(t;r))$, ($\underline{y}_k(t;r),\overline{y_k}(t;r),\underline{z}(t;r),\overline{z}(t;r)$)

Where

$$\underline{k_1}(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \min\left\{\begin{array}{l} h_k f(t_{k,n},u,\lambda_k(u_k)) \\ \backslash\, u \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \\ u_k \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \end{array}\right\},$$

$$\underline{l_1}(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \min\left\{\begin{array}{l} h_k f(t_{k,n},u,\lambda_k(u_k)) \\ \backslash\, u \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \\ u_k \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \end{array}\right\},$$

$$\overline{k_1}(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \max\left\{\begin{array}{l} h_k f(t_{k,n},u,\lambda_k(u_k)) \\ \backslash\, u \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \\ u_k \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \end{array}\right\},$$

$$\overline{l_1}(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \max\left\{\begin{array}{l} h_k f(t_{k,n},u,\lambda_k(u_k)) \\ \backslash\, u \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \\ u_k \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \end{array}\right\},$$

$$\underline{k_2}(t_{k,n}; y_{k,n}(r); z_{k,n}(r)) = \min\left\{\begin{array}{l} h_k f(t_{k,n} + \frac{1}{2}(h_k), u, \lambda_k(u_k)) \\ \backslash u \in \begin{bmatrix} \underline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \end{bmatrix} \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array}\right\},$$

$$\underline{l_2}(t_{k,n}; y_{k,n}(r); z_{k,n}(r)) = \min\left\{\begin{array}{l} h_k f(t_{k,n} + \frac{1}{2}(h_k), u, \lambda_k(u_k)) \\ \backslash u \in \begin{bmatrix} \underline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \end{bmatrix} \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array}\right\},$$

$$\overline{k}_2(t_{k,n}; y_{k,n}(r); z_{k,n}(r)) = \max\left\{\begin{array}{l} h_k f(t_{k,n} + \frac{1}{2}(h_k), u, \lambda_k(u_k)) \\ \backslash u \in \begin{bmatrix} \underline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \end{bmatrix} \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array}\right\},$$

$$\overline{l}_2(t_{k,n}; y_{k,n}(r); z_{k,n}(r)) = \max\left\{\begin{array}{l} h_k f(t_{k,n} + \frac{1}{2}(h_k), u, \lambda_k(u_k)) \\ \backslash u \in \begin{bmatrix} \underline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n}, y_{k,n}) \end{bmatrix} \\ u_k \in [\underline{y}_{k,0}(r), \overline{y}_{k,0}(r)] \end{array}\right\},$$

Like we can arrange

$\underline{k_3}(t_{k,n}; y_{k,n}(r); z_{k,n}(r))$, $\underline{l_3}(t_{k,n}; y_{k,n}(r); z_{k,n}(r))$, $\overline{k_3}(t_{k,n}; y_{k,n}(r); z_{k,n}(r))$, $\overline{l_3}(t_{k,n}; y_{k,n}(r); z_{k,n}(r))$,

$\underline{k_4}(t_{k,n}; y_{k,n}(r); z_{k,n}(r))$, $\underline{l_4}(t_{k,n}; y_{k,n}(r); z_{k,n}(r))$, $\overline{k_4}(t_{k,n}; y_{k,n}(r); z_{k,n}(r))$ & $\overline{l_4}(t_{k,n}; y_{k,n}(r); z_{k,n}(r))$.

Where

$$\underline{\Phi}_{k_1}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{1}{2}(\underline{k}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r)), \underline{l}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$

$$\overline{\Phi}_{k_1}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \overline{y}_{k,n}(r) + \frac{1}{2}(\overline{k}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r)), \overline{l}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$

$$\underline{\Phi}_{k_2}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \underline{y}_{k,n}(r) + \frac{1}{2}(\underline{k}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r)), \underline{l}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$

$$\overline{\Phi}_{k_2}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \overline{y}_{k,n}(r) + \frac{1}{2}(\overline{k}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r)), \overline{l}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$

$$\underline{\Phi}_{k_3}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \underline{y}_{k,n}(r) + (\underline{k}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r)), \underline{l}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$

$$\overline{\Phi}_{k_3}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \overline{y}_{k,n}(r) + (\overline{k}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r)), \overline{l}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$

Next we define:

$$S_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)] =$$

$$\frac{1}{6}\{\underline{k}_1(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + 2[\underline{k}_2(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + \underline{k}_3(t_{k,n}; y_{k,n}(r), z_{k,n}(r))] + \underline{k}_4(t_{k,n}; y_{k,n}(r), z_{k,n}(r))\}$$

$$T_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)] =$$

$$\frac{1}{6}\{\overline{k}_1(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + 2[\overline{k}_2(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + \overline{k}_3(t_{k,n}; y_{k,n}(r), z_{k,n}(r))] + \overline{k}_4(t_{k,n}; y_{k,n}(r), z_{k,n}(r))\}$$

$$S_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), z_{k,n}(r), \overline{z}_{k,n}(r)] =$$

$$\frac{1}{6}\{\underline{l}_1(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + 2[\underline{l}_2(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + l_3(t_{k,n}; y_{k,n}(r), z_{k,n}(r))] + \underline{l}_4(t_{k,n}; y_{k,n}(r), z_{k,n}(r))\}$$

$$T_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)] =$$

$$\frac{1}{6}\{\overline{l}_1(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + 2[\overline{l}_2(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + \overline{l}_3(t_{k,n}; y_{k,n}(r), z_{k,n}(r))] + \overline{l}_4(t_{k,n}; y_{k,n}(r), z_{k,n}(r))\}$$

The exact solution at $t_{k,n+1}$ is given by:

$$\begin{cases} F_{k.n+1}(r) = \underline{Y}_{k,n}(r) + S_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)], \\ G_{k,n+1}(r) = \overline{Y}_{k,n}(r) + T_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)]. \end{cases} \quad \ldots\ldots\ldots(4)$$

THE SIXTH ORDER RUNGE KUTTA FEHLBERG METHOD WITH HARMONIC MEAN

For a hybrid fuzzy fractional differential equation we develop the sixth order Runge Kutta Fehlberg method with harmonic mean when f and λ_k in (1) can be obtained via the Zadeh extension principle from:

f ϵ [R^+ X R X R, R] and $\lambda_k \epsilon$ C [R,R]

we assume that the existence and uniqueness of solutions of (1) hold for each [t_k, t_{k+1}]. For a fixed r, to integrate the system in (3) [t_0,t_1],[t_1,t_2],…..[t_k,t_{k+1}]……we replace each interval by a set of N_{k+1} discrete equally spaced grid points (including the end points) at which the exact solution x(t; r)=($\underline{x}(t;r), \overline{x}(t;r)$) is approximated by some ($\underline{y}(t;r), \overline{y}(t;r)$). For the chosen grid points on [t_k, t_{k+1}] at $t_{k,n} = t_k + nh_k$, $h_k = \frac{t_{k+1} - t_k}{N_k}$, $0 \le n \le N_k$.

Let ($\underline{Y}_k(t;r), \overline{Y_k}(t;r) \equiv$ ($\underline{x}_k(t;r), \overline{x_k}(t;r)$), ($\underline{y}_k(t;r), \overline{y_k}(t;r)$) and ($\underline{y}_k(t;r), \overline{y_k}(t;r)$) may be denoted respectively by ($\underline{Y}_{k,n}(t;r), \overline{Y}_{k,n}(t;r)$) and ($\underline{y}_{k,n}(t;r), \overline{y}_{k,n}(t;r)$).

We allow N_k's to vary over the [t_k, t_{k+1}]'s so that the h_k's may be comparable.

The Sixth Order Runge Kutta Fehlberg method for (1) is given by:

($\underline{Y}_k(t;r), \overline{Y_k}(t;r) \equiv$ ($\underline{x}_k(t;r), \overline{x_k}(t;r)$), ($\underline{y}_k(t;r), \overline{y_k}(t;r)$)

Where

$$\underline{k}_1(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \min\left\{\begin{matrix} h_k f(t_{k,n},u,\lambda_k(u_k)) \\ \backslash u \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \\ u_k \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \end{matrix}\right\},$$

$$\underline{l}_1(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \min\left\{\begin{matrix} h_k f(t_{k,n},u,\lambda_k(u_k)) \\ \backslash u \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \\ u_k \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \end{matrix}\right\},$$

$$\overline{k}_1(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \max\left\{\begin{matrix} h_k f(t_{k,n},u,\lambda_k(u_k)) \\ \backslash u \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \\ u_k \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \end{matrix}\right\},$$

$$\overline{l}_1(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \max\left\{\begin{matrix} h_k f(t_{k,n},u,\lambda_k(u_k)) \\ \backslash u \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \\ u_k \in \{[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)],[\underline{z}_{k,n}(r),\overline{z}_{k,n}(r)]\} \end{matrix}\right\},$$

$$\underline{k}_2(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \min\left\{\begin{matrix} h_k f(t_{k,n}+\frac{1}{2}(h_k),u,\lambda_k(u_k)) \\ \backslash u \in \begin{bmatrix} \underline{\Phi}_{k_1}(t_{k,n},y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n},y_{k,n}) \end{bmatrix} \\ u_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)] \end{matrix}\right\},$$

$$\underline{l}_2(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \min\left\{\begin{matrix} h_k f(t_{k,n}+\frac{1}{2}(h_k),u,\lambda_k(u_k)) \\ \backslash u \in \begin{bmatrix} \underline{\Phi}_{k_1}(t_{k,n},y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n},y_{k,n}) \end{bmatrix} \\ u_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)] \end{matrix}\right\},$$

$$\overline{k}_2(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \max\left\{\begin{matrix} h_k f(t_{k,n}+\frac{1}{2}(h_k),u,\lambda_k(u_k)) \\ \backslash u \in \begin{bmatrix} \underline{\Phi}_{k_1}(t_{k,n},y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n},y_{k,n}) \end{bmatrix} \\ u_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)] \end{matrix}\right\},$$

$$\bar{l}_2(t_{k,n};y_{k,n}(r);z_{k,n}(r)) = \max\left\{\begin{array}{l} h_k f(t_{k,n}+\frac{1}{2}(h_k),u,\lambda_k(u_k)) \\ \backslash u \in \begin{bmatrix} \underline{\Phi}_{k_1}(t_{k,n},y_{k,n}) \\ \overline{\Phi}_{k_1}(t_{k,n},y_{k,n}) \end{bmatrix} \\ u_k \in [\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)] \end{array}\right\},$$

Like we can arrange

$\underline{k}_3(t_{k,n};y_{k,n}(r);z_{k,n}(r))$, $\underline{l}_3(t_{k,n};y_{k,n}(r);z_{k,n}(r))$, $\overline{k}_3(t_{k,n};y_{k,n}(r);z_{k,n}(r))$, $\overline{l}_3(t_{k,n};y_{k,n}(r);z_{k,n}(r))$,
$\underline{k}_4(t_{k,n};y_{k,n}(r);z_{k,n}(r))$, $\underline{l}_4(t_{k,n};y_{k,n}(r);z_{k,n}(r))$, $\overline{k}_4(t_{k,n};y_{k,n}(r);z_{k,n}(r))$, $\overline{l}_4(t_{k,n};y_{k,n}(r);z_{k,n}(r))$,
$\overline{k}_5(t_{k,n};y_{k,n}(r);z_{k,n}(r))$, $\overline{l}_5(t_{k,n};y_{k,n}(r);z_{k,n}(r))$, $\overline{k}_6(t_{k,n};y_{k,n}(r);z_{k,n}(r))$ & $\overline{l}_6(t_{k,n};y_{k,n}(r);z_{k,n}(r))$

$$\underline{\Phi}_{k_1}(t_{k,n},y_{k,n}(r),z_{k,n}(r)) = \underline{f}(t_{k,n}+1/4*h,\underline{y}_{k,n}(r)+\underline{k}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)),\underline{z}_{k,n}(r)+\underline{l}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r))*h)$$

$$\overline{\Phi}_{k_1}(t_{k,n},y_{k,n}(r),z_{k,n}(r)) = \overline{f}(t_{k,n}+1/4*h,\overline{y}_{k,n}(r)+\overline{k}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)),\overline{z}_{k,n}(r)+\overline{l}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r))*h)$$

$$\underline{\Phi}_{k_2}(t_{k,n},y_{k,n}(r),z_{k,n}(r)) = \underline{f}(t_{k,n}+3/8*h,\underline{y}_{k,n}(r)+(3/32)*h*(\underline{k}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)+$$
$$3*\underline{k}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)),\underline{z}_{k,n}(r)+(3/32)*h*(\underline{l}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)+3*\underline{l}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)))$$
$$\overline{\Phi}_{k_2}(t_{k,n},y_{k,n}(r),z_{k,n}(r)) = \overline{f}(t_{k,n}+3/8*h,\overline{y}_{k,n}(r)+(3/32)*h*(\overline{k}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)+$$
$$3*\overline{k}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)),\overline{z}_{k,n}(r)+(3/32)*h*(\overline{l}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)+3*\overline{l}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)))$$
$$\underline{\Phi}_{k_3}(t_{k,n},y_{k,n}(r),z_{k,n}(r)) = \underline{f}(t_{k,n}+(12/13)*h,\underline{y}_{k,n}(r)+(12/2197)*h*$$
$$(161*\underline{k}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)-600*\underline{k}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)+608\underline{k}_3(t_{k,n},y_{k,n}(r),z_{k,n}(r)),$$
$$\underline{z}_{k,n}(r)+(12/2197)*h*(161*\underline{l}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)-600*\underline{l}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)$$
$$+608\underline{l}_3(t_{k,n},y_{k,n}(r),z_{k,n}(r)))$$
$$\overline{\Phi}_{k_3}(t_{k,n},y_{k,n}(r),z_{k,n}(r)) = \overline{f}(t_{k,n}+(12/13)*h,\overline{y}_{k,n}(r)+(12/2197)*h*$$
$$(161*\overline{k}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)-600*\overline{k}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)+608\overline{k}_3(t_{k,n},y_{k,n}(r),z_{k,n}(r)),$$
$$\overline{z}_{k,n}(r)+(12/2197)*h*(161*\overline{l}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)-600*\overline{l}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)$$
$$+608\overline{l}_3(t_{k,n},y_{k,n}(r),z_{k,n}(r)))$$
$$\underline{\Phi}_{k_4}(t_{k,n},y_{k,n}(r),z_{k,n}(r)) = \underline{f}(t_{k,n}+h,\underline{y}_{k,n}(r)+(1/4104)*h*(8341*\underline{k}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)$$
$$-32832*\underline{k}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)+29440*\underline{k}_3(t_{k,n},y_{k,n}(r),z_{k,n}(r)-845*\underline{k}_4(t_{k,n},y_{k,n}(r),z_{k,n}(r)),$$
$$\underline{z}_{k,n}(r)+(1/4104)*h*(8341*\underline{l}_1(t_{k,n},y_{k,n}(r),z_{k,n}(r)-32832*\underline{l}_2(t_{k,n},y_{k,n}(r),z_{k,n}(r)+$$
$$29440*\underline{l}_3(t_{k,n},y_{k,n}(r),z_{k,n}(r)-845*\underline{l}_4(t_{k,n},y_{k,n}(r),z_{k,n}(r)))$$

$$\overline{\Phi}_{k_4}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \overline{f}(t_{k,n} + h, \overline{y}_{k,n}(r) + (1/4104)*h*(8341*\overline{k}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r)$$
$$-32832*\overline{k}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r) + 29440*\overline{k}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - 845*\overline{k}_4(t_{k,n}, y_{k,n}(r), z_{k,n}(r)),$$
$$\overline{z}_{k,n}(r) + (1/4104)*h*(8341*\overline{l}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - 32832*\overline{l}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r) +$$
$$29440*\overline{l}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - 845*\overline{l}_4(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$
$$\underline{\Phi}_{k_5}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \underline{f}(t_{k,n} + (0.5)h, \underline{y}_{k,n}(r) + h*(-(8/27)*\underline{k}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r)$$
$$+2*\underline{k}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - (3544/2565)*\underline{k}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r) + (1859/4104)$$
$$*\underline{k}_4(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - (11/40)*\underline{k}_5(t_{k,n}, y_{k,n}(r), z_{k,n}(r)), \underline{z}_{k,n}(r)$$
$$+h*(-(8/27)*\underline{l}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r) + 2*\underline{l}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - (3544/2565)*$$
$$\underline{l}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r) + (1859/4104)*\underline{l}_4(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - (11/40)*\underline{l}_5(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$
$$\overline{\Phi}_{k_5}(t_{k,n}, y_{k,n}(r), z_{k,n}(r)) = \overline{f}(t_{k,n} + (0.5)h, \overline{y}_{k,n}(r) + h*(-(8/27)*\overline{k}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r)$$
$$+2*\overline{k}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - (3544/2565)*\overline{k}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r) + (1859/4104)$$
$$*\overline{k}_4(t_{k,n}, y_{k,n}(r), z_{k,n}(r) - (11/40)*\overline{k}_5(t_{k,n}, y_{k,n}(r), z_{k,n}(r)),$$
$$\overline{z}_{k,n}(r) + h*(-(8/27)*\overline{l}_1(t_{k,n}, y_{k,n}(r), z_{k,n}(r) + 2*\overline{l}_2(t_{k,n}, y_{k,n}(r), z_{k,n}(r) -$$
$$(3544/2565)*\overline{l}_3(t_{k,n}, y_{k,n}(r), z_{k,n}(r) + (1859/4104)*\overline{l}_4(t_{k,n}, y_{k,n}(r), z_{k,n}(r) -$$
$$(11/40)*\overline{l}_5(t_{k,n}, y_{k,n}(r), z_{k,n}(r)))$$

Next we define:

$$S_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)]$$
$$= \frac{h}{5}\{(16/27)\underline{k}_1(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + (6656/2565)\underline{k}_3(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + (28561/11286)$$
$$\underline{k}_4(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) - (9/10)\underline{k}_5(t_{k,n}; y_{k,n}(r), z_{k,n}(r) + (12/11)\underline{k}_6(t_{k,n}; y_{k,n}(r), z_{k,n}(r))\}$$
$$T_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)]$$
$$= \frac{h}{5}\{(16/27)\overline{k}_1(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + (6656/2565)\overline{k}_3(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) + (28561/11286)$$
$$\overline{k}_4(t_{k,n}; y_{k,n}(r), z_{k,n}(r)) - (9/10)\overline{k}_5(t_{k,n}; y_{k,n}(r), z_{k,n}(r) + (12/11)\overline{k}_6(t_{k,n}; y_{k,n}(r), z_{k,n}(r))\}$$

The exact solution at $t_{k,n+1}$ is given by:

$$\begin{cases} F_{k.n+1}(r) = \underline{Y}_{k,n}(r) + S_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)], \\ G_{k,n+1}(r) = \overline{Y}_{k,n}(r) + T_k[t_{k,n,} \underline{y}_{k,n}(r), \overline{y}_{k,n}(r), \underline{z}_{k,n}(r), \overline{z}_{k,n}(r)]. \end{cases} \quad \text{.........(5)}$$

Degree of Sub Element hood:

Let X be a Universal, U be a set of parameters and let ($F_{k,n+1}$) and ($G_{k,n+1}$) are two fuzzy elements of X. Then the degree of sub element hood denoted by

S($F_{k,n+1}$, $G_{k,n+1}$) is defined as,

$$\mathfrak{S}(F_{k,n+1}, G_{k,n+1}) = \frac{1}{\left|(F_{k,n+1})\right|}\left\{\left|(F_{k,n+1})\right| - \sum \max\{0, \left(F_{k,n+1}\right) - \left(G_{k,n+1}\right)\}\right\}$$

Where $\left|(F_{k,n+1})\right| = \sum e_j \in A\exp(F_{k,n+1})$

IV. ERROR ANALYSIS

In this section, we present the example for analysing the error of the hybrid fuzzy fractional differential equations between Runge Kutta 4th order & 6th order Fehlberg Method.
Consider the following second order hybrid fuzzy fractional differential equation:

$${}_cD_a^{\beta}\ X\ (t) = Z \quad \& \quad {}_cD_a^{\beta}\ Z\ (t) = XZ^2 - Y^2 \quad \ldots\ldots (6)$$
$$X(0) = X_0,$$

where $\beta \in (0,1]$, t > 0, and X_0 is any triangular fuzzy number.
This problem is a generalization of the following hybrid fuzzy fractional differential equation:
We can find the solution of the hybrid fractional fuzzy differential equation, by the method of Runge Kutta 4th order & Fehlberg 6th order Methods. We compared & generalized the error of the hybrid fractional fuzzy differential equation, also we illustrated the figure and in the table for this generalization by using Matlab.

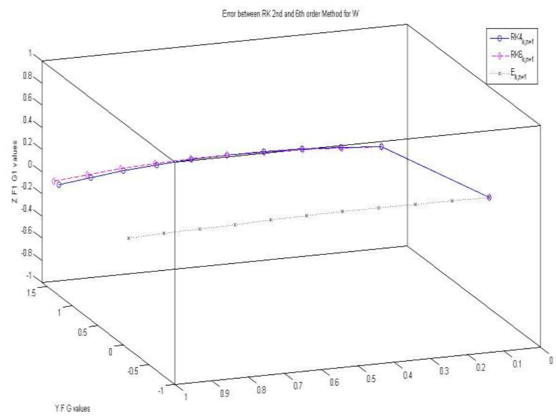

Fig.1. Error Analysis for Y values of the Example

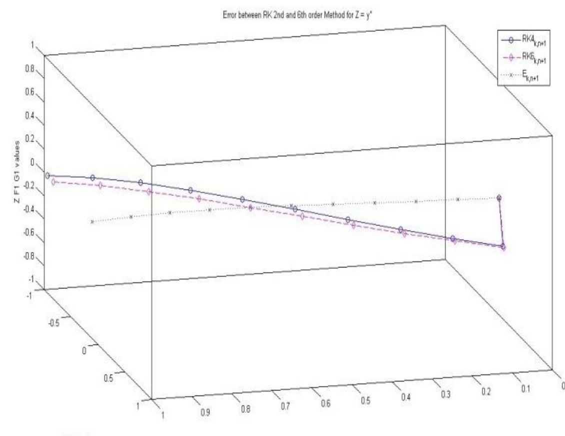

Fig.2. Error Analysis for Z values of the Exampl

S.No	T	RK 4th order (W)	RK 6th order (W)	Error (W)	RK 4th order (Z)	RK 6th order (Z)	Error (Z)
1	0	0	0	0	1.3	1.3	0
2	0.1	1.391266317	1.38252921	0.008737	0.822369471	0.843479358	-0.02111
3	0.2	1.463836926	1.45043447	0.013402	0.625779615	0.672041993	-0.04626
4	0.3	1.515773725	1.50232107	0.013453	0.409924119	0.484592947	-0.07467
5	0.4	1.545284394	1.53679615	0.008488	0.178164727	0.2823831	-0.10422
6	0.5	1.551126301	1.55270406	-0.00158	-0.06180337	0.069581679	-0.13139
7	0.6	1.533042361	1.54940226	-0.01636	-0.29809153	-0.14644294	-0.15165
8	0.7	1.492116438	1.52703428	-0.03492	-0.51615286	-0.35553489	-0.16062
9	0.8	1.430901028	1.48672389	-0.05582	-0.70172741	-0.54612954	-0.1556
10	0.9	1.353215407	1.43060781	-0.07739	-0.84426266	-0.70745571	-0.13681
11	1.0	1.263641534	1.36166109	-0.09802	-0.93929549	-0.8318705	-0.10742

Table.1. Error Analysis for Y & Z values of the Example

V. CONCLUSION AND FUTURE SCOPE

In this paper, we have discussed the error analysis of the hybrid fuzzy fractional differential equation by Runge Kutta 4^{th} order method & 6^{th} order Fehlberg method. Final results showed that the solution of hybrid fuzzy fractional differential equations approaches the solution of hybrid fuzzy differential equations as the fractional order approaches the integer order. The results of the study reveal that the proposed error analysis method with fuzzy fractional derivatives is efficient, accurate, and convenient for solving the hybrid fuzzy fractional differential equations. We can develop the error analysis for higher order system.

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