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Transmuted Generalized Uniform Distribution

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Abstract-In this paper we have introduced a new probability model called as transmuted generalized Uniform distribution (TGUD) by using the quadratic rank transmutation map studied by Shaw and Buckley (2007). The moments, survival function, failure rate and reverse hazard rate of the distribution have been derived. The parameters have been estimated by the maximum likelihood method. Also we have obtained the pdf of r^{th}, 1^{st} and n^{th} order statistics.

Keywords: *Transmuted generalized Uniform distribution, Moments, Reliability analysis, Parameter estimation, Order statistics*

I. INTRODUCTION

Statistics is the science of drawing inferences about random phenomena in which chances play an important role. Transmuted distributions have been discussed dynamically in frequently occurring large scale experimental statistical data for model selection and related issues. A significant progress has been made towards the generalization of some well known distributions. These extended distributions find their application in many lifetime problems like medical, economics, finance, environmental, engineering and biomedical sciences. There are several distributions which can be used to model such kind of experimental data. The procedures used in such a statistical analysis depend heavily on the assumed probability model or distributions. That is why the development of large classes of standard probability distributions along with relevant statistical methodologies has been expanded. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models. Shaw and Buckley (2007) introduced the new quadratic rank transmutation map (QRTM) technique. The quadratic rank transmutation map, will be used in this paper to derive a generalization of the Generalized Uniform distribution. This generalization is termed as Transmuted Generalized Uniform distribution (TGUD). Generally, transmutation maps are a convenient way of constructing new distributions, in particular, survival ones. Aryal and Tsokos (2009) discussed on the transmuted extreme value distribution. Afify *et al.* (2015) discussed transmuted weibull lomax distribution. Deepshikha *et al*. (2017) also discussed on transmuted exponential gumbel distribution and its applications to water quality data. Recently, Rather and Subramanian (2018) obtained a new transmuted mukherjee-islam failure model which shows more flexibility than classical distributions.

A probability distribution can be characterized through various methods. Generalized uniform distribution is characterized through the conditional expectation of lower record values. Ali *et al.* (2007), study a new property exponentiated generalized uniform distribution. Bhatt (2014), discussed characterization of generalized uniform distribution through expectation. Khan and khan (2017), obtained the characterization of generalized uniform distribution based on lower record values.

According to quadratic rank transmutation map (QRTM) technique approach, a random variable X is said to have a Transmuted distribution, if its cdf is given by

$$F(x) = [(1+\lambda)G(x) - \lambda\left(G(x)\right)^2] \qquad |\lambda| \le 1 \qquad (1)$$

Where $G(x)$ is the cdf of the base distribution and $F(x)$ is the cdf of Transmuted distribution.
Differentiating (1) with respect to x, we will get the pdf of Transmuted distribution as

$$f(x) = g(x)\left(1+\lambda-2\lambda G(x)\right) \qquad (2)$$

where *f(x)* and *g(x)* are the corresponding probability density functions of *F(x)* and *G(x)* respectively.

II. TRANSMUTED GENERALIZED UNIFORM DISTRIBUTION

Consider the Generalized Uniform distribution with probability density function (pdf) as

$$g(x;\alpha,\beta)=\left(\frac{\alpha+1}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha} \qquad 0<x<\beta,\ -1<\alpha \qquad (3)$$

and its cumulative distribution function (cdf) is

$$G(x;\alpha,\beta)=\left(\frac{x}{\beta}\right)^{\alpha+1} \qquad (4)$$

Where α is the shape parameter and β is the scale parameter.
Using (4) in (1), we will get the cdf of transmuted generalized Uniform distribution (TGUD)

$$F(x;\alpha,\beta,\lambda)=\left(\frac{x}{\beta}\right)^{\alpha+1}\left[(1+\lambda)-\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right] \qquad 0<x<\beta,\ -1<\alpha \qquad (5)$$

Also substitute the values of (3) and (4) in equation (2), we will get the pdf of transmuted generalized Uniform distribution

$$f(x;\alpha,\beta,\lambda)=\left(\frac{\alpha+1}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha}\left[(1+\lambda)-2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right] \qquad (6)$$

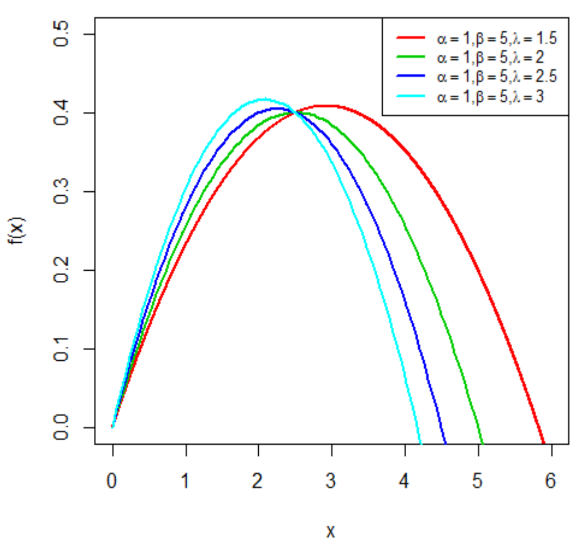

Fig.1: pdf plot of Transmuted Generalized Uniform Distribution

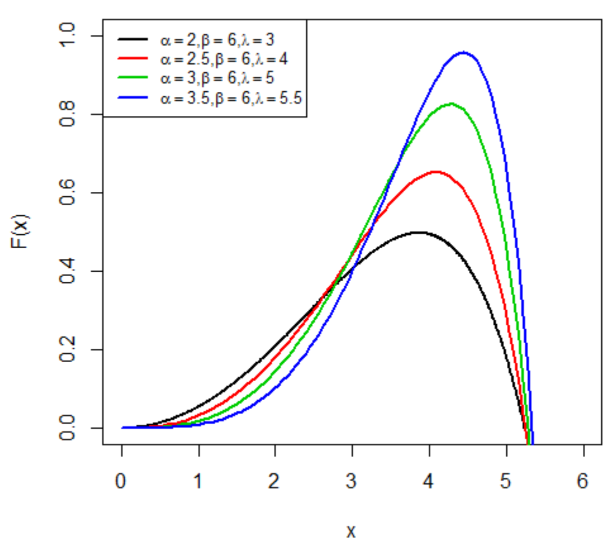

Fig.2: Cdf plot of Transmuted Generalized Uniform distribution

III. RELIABILITY ANALYSIS

In this section, we will discuss the reliability function, hazard rate function and reverse hazard rate function for the transmuted generalized uniform distribution.

The probability that a system survives beyond a specified time is known as reliability function or survivor or survival function and is given by

$$R(x) = 1 - F(x)$$

$$R(x) = 1 - \left(\frac{x}{\beta}\right)^{\alpha+1}\left[(1+\lambda) - \lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right]$$

The hazard function is also called as hazard rate or failure rate and is given by

$$h(x) = \frac{f(x)}{1-F(x)}$$

$$h(x) = \frac{\left(\frac{\alpha+1}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha}\left[(1+\lambda) - 2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right]}{1 - \left(\frac{x}{\beta}\right)^{\alpha+1}\left[(1+\lambda) - \lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right]}$$

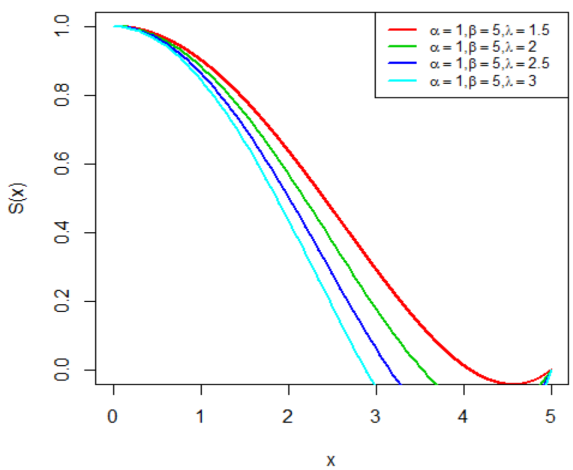

Fig.3: Showing the Srvival function curves of TGUD

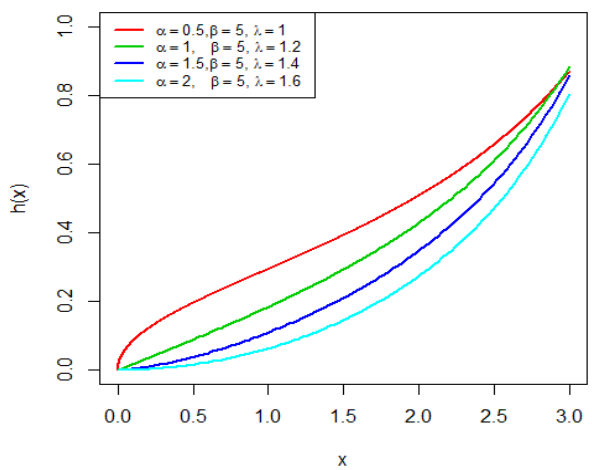

Fig.4: Showing the hazard rate curves of TGUD

and the reverse hazard function for the transmuted generalized Uniform distribution is given by

$$h_r(x) = \frac{f(x)}{F(x)}$$

$$h_r(x) = \frac{\left(\frac{\alpha+1}{\beta}\right)\left[(1+\lambda) - 2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right]}{\left(\frac{x}{\beta}\right)\left[(1+\lambda) - \lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right]}$$

IV. MOMENTS

Let X denotes the random variable of transmuted generalized Uniform distribution, then the r-th order moment $E(X^r)$ of transmuted generalized Uniform distribution can be obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_l(x;\theta)dx$$

$$= \int_0^\beta x^r \left(\frac{\alpha+1}{\beta}\right)\left(\frac{x}{\beta}\right)^a \left[(1+\lambda) - 2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right]dx$$

$$= (1+\lambda)\left(\frac{\alpha+1}{\beta^{\alpha+1}}\right)\int_0^\beta x^{\alpha+r}\,dx - 2\lambda\left(\frac{\alpha+1}{\beta^{2\alpha+2}}\right)\int_0^\beta x^{r+2\alpha+1}dx$$

$$= \frac{(1+\lambda)(\alpha+1)\beta^r}{(\alpha+r+1)} - \frac{2\lambda(\alpha+1)\beta^r}{(2\alpha+r+2)}$$

Thus, the r^{th} order moment of TGUD is obtained

$$E(X^r) = \mu_r' = \frac{\beta^r(\alpha+1)[2(\alpha+1)+r(1-\lambda)]}{(\alpha+r+1)(2\alpha+r+2)}$$

Put r =1,2,3,4 we will obtain the first four moments

$$\text{Mean} = \mu_1' = \frac{\beta(\alpha+1)(2\alpha-\lambda+3)}{(\alpha+2)(2\alpha+3)}$$

$$\mu_2' = \frac{\beta^2(\alpha+1)(\alpha-\lambda+2)}{(\alpha+3)(\alpha+2)}$$

$$\text{variance} = \frac{\beta^2(\alpha+1)[\alpha\lambda(3+2\alpha)-\alpha\lambda^2(\alpha+4)-28\alpha(\alpha+3)-3(\lambda^2+21)]}{(\alpha+3)(\alpha+2)^2(2\alpha+3)^2}$$

$$\text{Standard Deviation}(\sigma) = \frac{\beta}{(\alpha+2)(2\alpha+3)} \times \sqrt{\frac{(\alpha+1)}{(\alpha+3)}\left(\alpha\lambda(3+2\alpha)-\alpha\lambda^2(\alpha+4)-28\alpha(\alpha+3)-3(\lambda^2+21)\right)}$$

$$\text{Coefficient of Variation (C.V.)} = \frac{\sigma}{\mu_1'} = \frac{1}{(\alpha+1)(2\alpha-\lambda+3)} \times \sqrt{\frac{(\alpha+1)}{(\alpha+3)}\left(\alpha\lambda(3+2\alpha)-\alpha\lambda^2(\alpha+4)-28\alpha(\alpha+3)-3(\lambda^2+21)\right)}$$

$$\text{Coefficient of Dispersion}(\gamma) = \frac{\sigma^2}{\mu_1'} = \frac{\beta\left(\alpha\lambda(3+2\alpha)-\alpha\lambda^2(\alpha+4)-28\alpha(\alpha+3)-3(\lambda^2+21)\right)}{(\alpha+2)(\alpha+3)(2\alpha+3)(2\alpha-\lambda+3)}$$

V. MOMENT GENERATING FUNCTION AND CHARACTERISTIC FUNCTION OF TGUD

Let *X* have a transmuted generalized Uniform distribution, then the moment generating function of *X* is given by

$$M_x(t) = E(e^{tx}) = \int_0^{\beta} e^{tx} f(x;\alpha,\beta,\lambda)dx$$

Using Taylor's series

$$M_x(t) = E(e^{tx}) = \int_0^{\beta} \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f(x)dx$$

$$= \int_0^{\beta} \sum_{j=0}^{\infty} \frac{t^j}{j!} x^j f(x)dx$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j^{'}$$

$$\Rightarrow M_x(t) = (\alpha+1)\sum_{j=0}^{\infty} \frac{t^j \beta^j}{j!}\left(\frac{[(1+\lambda)(2\alpha+j+2) - 2\lambda(\alpha+j+1)]}{(\alpha+j+1)(2\alpha+j+2)}\right)$$

Similarly, the characteristic function of TGUD can be obtained in a similar way

$$\varphi_x(t) = M_x(it)$$

$$\Rightarrow \varphi_x(t) = (\alpha+1)\sum_{j=0}^{\infty} \frac{(it)^j \beta^j}{j!}\left(\frac{[(1+\lambda)(2\alpha+j+2) - 2\lambda(\alpha+j+1)]}{(\alpha+j+1)(2\alpha+j+2)}\right)$$

VI. RANDOM NUMBER GENERATION

By using the Inversion method, we can easily generate the random numbers from the transmuted generalized uniform distribution as given by

$$(1+\lambda)\left(\frac{x}{\beta}\right)^{\alpha+1} - \lambda\left(\frac{x}{\beta}\right)^{2\alpha+2} = u$$

Where u ~ U (0, 1), after simplification we get

$$\lambda\left(\left(\frac{x}{\beta}\right)^{\alpha+1}\right)^2 - (1+\lambda)\left(\frac{x}{\beta}\right)^{\alpha+1} + u = 0$$

$$\Rightarrow x = \beta\left(\frac{(1+\lambda) + \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}\right)^{\frac{1}{\alpha+1}} \quad (7)$$

When the parameters *α, β,* and *λ* are known, we can use equation (7) to generate random numbers.

VII. PARAMETER ESTIMATION

In this section, we will use maximum likelihood method for estimating the parameters of transmuted generalized uniform distribution. Let $X_1, X_2, ..., X_n$ denotes the random sample of size n from the transmuted generalized uniform distribution. Then the likelihood function is given by

$$L(\alpha,\beta,\lambda)=\frac{(\alpha+1)^n}{\beta^{n(\alpha+1)}}\prod_{i=1}^{n}x_i^{\alpha}\prod_{i=1}^{n}\left((1+\lambda)-2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right)$$

The log-likelihood function comes out to be

$$\log L(\alpha,\beta,\lambda)=n\log(\alpha+1)-n(\alpha+1)\log\beta+\alpha\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log\left((1+\lambda)-2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right) \quad (8)$$

Therefore, the maximum likelihood estimator of α, β, λ which maximize equation (8), must satisfy the following normal equations given by

$$\frac{\partial}{\partial\alpha}\log L(\alpha,\beta,\lambda)=\frac{n}{\alpha+1}-n\log\beta+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\left(\frac{-2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha+1}\log\left(\frac{x_i}{\beta}\right)}{(1+\lambda)-2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha+1}}\right)=0 \quad (9)$$

$$\frac{\partial}{\partial\beta}\log L(\alpha,\beta,\lambda)=-\frac{n\alpha}{\beta}+\sum_{i=1}^{n}\left(\frac{2\lambda\left(\frac{x_i}{\beta^2}\right)^{\alpha+1}}{(1+\lambda)-2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha+1}}\right)=0 \quad (10)$$

$$\frac{\partial}{\partial\lambda}\log L(\alpha,\beta,\lambda)=\sum_{i=1}^{n}\left(\frac{1-2\left(\frac{x_i}{\beta^2}\right)^{\alpha+1}}{(1+\lambda)-2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha+1}}\right)=0 \quad (11)$$

The likelihood equations (9), (10), (11) are in the complicated form. Algebraically, it is very difficult to solve the solution of the normal equations. Usually, it is more convenient if we use non-linear optimization algorithms such as Newton-Raphson or Quasi-Newton algorithms to maximize numerically the log likelihood function (8).

8. ORDER STATISTICS

Order statistics plays an important role in many theoretical as well as practical fields.

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from the continuous population with probability density function $f_x(x)$ and cumulative density function $F_x(x)$, then the pdf of r^{th} order statistics $X_{(r)}$ can be written as

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left[F_X(x)\right]^{r-1}\left[1-F_X(x)\right]^{n-r} \qquad (12)$$

Substitute the values of (5) and (6) in equation (12), we will get the pdf of r^{th} order statistics $X_{(r)}$ for transmuted generalized uniform distribution and is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\left(\frac{\alpha+1}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha}\left((1+\lambda)-2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right)\right)$$

$$\times\left(\left(\frac{x}{\beta}\right)^{\alpha+1}\left((1+\lambda)-\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right)\right)^{r-1}\left(1-\left(\frac{x}{\beta}\right)^{\alpha+1}\left((1+\lambda)-\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right)\right)^{n-r} \qquad (13)$$

From equation (13), the density function of largest order statistics $X_{(n)}$ is given by

$$f_{X(n)}(x) = n\left(\left(\frac{\alpha+1}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha}\left((1+\lambda)-2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right)\right)\left(\left(\frac{x}{\beta}\right)^{\alpha+1}\left((1+\lambda)-\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right)\right)^{n-1}$$

and the pdf of first order statistics $X_{(1)}$ is given by

$$f_{X(1)}(x) = n\left(\left(\frac{\alpha+1}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha}\left((1+\lambda)-2\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right)\right)\left(1-\left(\frac{x}{\beta}\right)^{\alpha+1}\left((1+\lambda)-\lambda\left(\frac{x}{\beta}\right)^{\alpha+1}\right)\right)^{n-1}$$

9. CONCLUSION

In this paper, we have studied a new three parametric distribution called as the Transmuted Generalized Uniform distribution. The subject distribution is obtained by using quadratic rank transmutation map and taking the generalized uniform distribution as base distribution. Its a three parametric distribution with two scale and one shape parameter. By using certain special functions, its mathematical properties, moments, failure rate, survival function, reverse hazard rate have been obtained. The parameters have been estimated by using the maximum likelihood estimator.

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