Transmuted Zeghdoudi Distribution: Theory and Applications

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Abstract- The Transmuted Zeghdoudi distribution (TZD) is derived using the Quadratic Rank Transmutation Map (QRTM) approach. Some mathematical properties of the proposed distribution are derived such as moments, skewness, kurtosis, mean residual lifetime, hazard rate, order statistics and reliability analysis etc. Estimation Properties like maximum likelihood estimation procedure is derived to obtain the estimates of parameters. A data set are analyzed to illustrate its applicability and to compare that this distribution performs better than many other distributions.

Keywords—Zeghdoudi Distribution, Quadratic Rank Transmutation Map, Moment Generating Function, Order Statistics, Maximum Likelihood Estimation

I. Introduction

Several Statistical distributions plays a very important role in prediction and analysis of real lifetime data. Data obtained from various fields such as financial, biological, physical, actuarial, engineering are fitted with the statistical distribution for prediction and analysis. Some well-known continuous distribution such as Exponential, Gamma, Lindley, Weibull, that have been applied to model the lifetime data sets. For the last 30 years, several new distributions were introduced that has a wide range of application in the field of data modelling. Due to the use of sophisticated technology, more and more data are obtained from various sources. So, research is going to develop new distribution that fits the data better and gives much more reliable estimates. Some of them one parameter mixture distribution such as Beta-Pareto[1], Lindley[2], Beta-Laplace[3], Xgamma[4], Weibull- Pareto[5], Beta Exponentiated-Pareto[6], Transmuted Pranav Distribution[7], Transmuted Lindley Distribution[8], and Gamma-Lindley Distribution[9] and so on were used to fit the lifetime data better and have high flexibility. The Exponential families of distributions have played a vital role in probability distribution for modelling the real world phenomenon.

Recently, on parameter exponential families of distribution which is the mixture of $\text{Gamma}(2, \theta)$ and $\text{Gamma}(3, \theta)$ with mixing probability $\theta/(2 + \theta)$ have been introduced and named as Zeghdoudi distribution[10]. The probability density function (pdf) of Zeghdoudi distribution is defined as

$$g(x; \theta) = \frac{\theta^3 x(1+x)}{2+\theta} e^{-\theta x} \quad x > 0, \theta > 0 \quad (1)$$

and its cumulative distribution function (cdf) is given by

$$G(x) = 1 - \frac{\theta^2 x^2 + \theta(\theta+2)x + \theta + 2}{2+\theta} e^{-\theta x} \quad x > 0, \theta > 0. \quad (2)$$

We use the transmutation map approach which was suggested by Shaw et. al.[11]

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x), \quad |\lambda| \leq 1, \quad (3)$$

where $G(x)$ is the cdf of the base distribution.

When $\lambda = 0$ we have the distribution of the Zeghdoudi random variable. In this paper, we have introduced a transmuted version of the Zeghdoudi Distribution using Quadratic Rank Transmutation Map (QRTM) approach. The uses of this transmutation approach have played a vital role in modeling the lifetime data. We named the distribution as Transmuted Zeghdoudi Distribution (TZD).

The paper is organized as follows, Section II contains the introduction of pdf and cdf of Transmuted Zeghdoudi Distribution. Section III contains the mathematical properties of the distribution such as moments, moment generating function and second, third and fourth order moment. Section IV contains the measures of skewness and kurtosis and plotted for different values. Section V contains the order statistics. Section VI contains Reliability analysis and Hazard

Function, Section VII contains simulation and parameter estimation using Maximum Likelihood Estimation., Section VIII contains the application to two real lifetime data and compare with two other distributions using related measures. Section IX concludes research work.

II. Transmuted Zeghdoudi distribution

Definition 4.1 A random variable X is said to have the transmuted Zeghdoudi distribution with parameter θ and λ if its probability density is defined as

$$f(x) = \frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta} \left[(1-\lambda) + 2\lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \quad ; \quad \theta, x > 0, \quad (4)$$

Figure 1: Plot of p.d.f of Transmuted Zeghdoudi Distribution

and its cdf is given by

$$F(x) = \left[1 - \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \left[1 + \lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \quad ; \quad \theta, x > 0. \quad (5)$$

Figure 2: Plot of c.d.f of Transmuted Zeghdoudi Distribution

The transmuted Zeghdoudi distribution is an extended model of Gamma distribution to analyse the more complex lifetime data.

III. Moments and quantiles

Let $x \sim \text{TG}(\theta, \lambda)$, then the r^{th} raw moment about the origin is given by

$$E(X^k) = \int_0^\infty x^k \frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta} \left[(1-\lambda) + 2\lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] dx. \quad (6)$$

And the first, second, third and fourth order moment about origin are given by:

$$\begin{aligned}\mu_1' &= \frac{\theta^3}{2+\theta} \cdot \left[\frac{2}{\theta^3} \left(1 + \frac{3}{\theta} \right) - \frac{2\lambda}{\theta^3} \left(1 + \frac{3}{\theta} \right) + \frac{48\lambda\theta^2}{(\theta+2)(2\theta)^5} \left(1 + \frac{5}{2\theta} \right) + \frac{12\lambda\theta}{(2\theta)^4} \left(1 + \frac{2}{\theta} \right) + \frac{4\lambda}{(2\theta)^3} \left(1 + \frac{3}{2\theta} \right) \right], \\ \mu_2' &= \frac{\theta^3}{2+\theta} \cdot \left[\frac{6}{\theta^4} \left(1 + \frac{4}{\theta} \right) - \frac{6\lambda}{\theta^4} \left(1 + \frac{4}{\theta} \right) + \frac{240\lambda\theta^2}{(\theta+2)(2\theta)^6} \left(1 + \frac{3}{\theta} \right) + \frac{48\lambda\theta}{(2\theta)^5} \left(1 + \frac{5}{2\theta} \right) + \frac{12\lambda}{(2\theta)^4} \left(1 + \frac{2}{\theta} \right) \right], \\ \mu_3' &= \frac{\theta^3}{2+\theta} \cdot \left[\frac{24}{\theta^5} \left(1 + \frac{5}{\theta} \right) - \frac{24\lambda}{\theta^5} \left(1 + \frac{5}{\theta} \right) + \frac{1440\lambda\theta^2}{(\theta+2)(2\theta)^7} \left(1 + \frac{7}{2\theta} \right) + \frac{240\lambda\theta}{(2\theta)^6} \left(1 + \frac{3}{\theta} \right) + \frac{48\lambda}{(2\theta)^5} \left(1 + \frac{5}{2\theta} \right) \right], \\ \mu_4' &= \frac{\theta^3}{2+\theta} \cdot \left[\frac{120}{\theta^6} \left(1 + \frac{6}{\theta} \right) - \frac{120\lambda}{\theta^6} \left(1 + \frac{6}{\theta} \right) + \frac{10080\lambda\theta^2}{(\theta+2)(2\theta)^8} \left(1 + \frac{4}{\theta} \right) + \frac{1400\lambda\theta}{(2\theta)^7} \left(1 + \frac{7}{2\theta} \right) + \frac{240\lambda}{(2\theta)^6} \left(1 + \frac{3}{\theta} \right) \right].\end{aligned}$$

As variance, skewness & kurtosis calculative expressions are so complicate, therefore the graph of skewness & kurtosis is plotted.

Figure 3: The graph of skewness and kurtosis for λ from -1 to +1.

IV. Moment Generating Function

The moment generating function in terms of the moment are given by

$$\begin{aligned}M_X(t) &= E(e^{tx}) = \int_0^\infty e^{tx} \frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta} \left[(1-\lambda) + 2\lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] dx \\ &= \frac{\theta^3}{2+\theta} \left[\left(\frac{1-\lambda}{(\theta-t)^2} \right) \left(1 + \frac{2}{\theta-t} \right) + \frac{12\lambda\theta^2}{(2+\theta)(2\theta-t)^4} \left(1 + \frac{4}{2\theta-t} \right) \right. \\ &\quad \left. + \frac{4\lambda\theta}{(2\theta-t)^3} \left(1 + \frac{3}{2\theta-t} \right) + \frac{2\lambda}{(2\theta-t)^2} \left(1 + \frac{2}{(2\theta-t)^3} \right) \right].\end{aligned}$$

V. Order Statistics

Let $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ be the ordered random sample drawn from the Transmuted Zeghdoudi Distribution, then the c.d.f. and p.d.f. of the r^{th} order statistics is given by

$$G_r(y) = \sum_{i=r}^n n_i [G(y; \theta; \lambda)]^i [1 - G(y; \theta; \lambda)]^{n-i},$$

$$g_r(y) = \frac{n!}{(r-1)!(n-r)!} [G(y; \theta; \lambda)]^{r-1} [1 - G(y; \theta; \lambda)]^{n-r} g(y; \theta; \lambda).$$

So, the p.d.f. and c.d.f. of $y_{(1)}$ and $y_{(n)}$ is given by

$$g_{y_1}(y) = n \left[\frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta} \left((1-\lambda) + 2\lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right) \right] \left[1 - \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \left[1 + \lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right]^{n-1},$$

$$g_{y_n}(y) = n \left[\frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta} \left((1-\lambda) + 2\lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right) \right] \left[1 - \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \left[1 + \lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right]^{n-1},$$

$$G_{y_1}(y) = 1 - \left(\left[1 - \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \left[1 + \lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \right)^{n-1},$$

$$G_{y_n}(y) = \left(\left[1 - \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \left[1 + \lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \right)^n.$$

VI. Survival Properties

The hazard rate function or failure rate function for a continuous distribution with pdf $f(x)$, cdf $F(x)$ and survival function (sf) $S(x)$ is defined as,

$$h(x) = \ln \lim_{\Delta x \rightarrow 0} \frac{P(X < x + \Delta x | X > x)}{\Delta x} = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)}.$$

For the transmuted zeghdoudi distribution, the hazard rate function is given by

$$h(x) = \frac{\frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta} \left[(1-\lambda) + 2\lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right]}{1 - \left[1 - \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right] \left[1 + \lambda \left(\frac{\theta^2 x^2}{2+\theta} + \theta x + 1 \right) e^{-\theta x} \right]}$$

Figure 4: Plot of hazard function of Transmuted zeghdoudi distribution.

VII. Estimation of the parameters

Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the pdf in (4). The log likelihood equation is as follows

$$L(\theta, \lambda) = 3n \log(\theta) - n \log(2 + \theta) + \sum_{i=1}^n \log[x_i(x_i + 1)] - n\theta \bar{x} + \sum_{i=1}^n \log \left(1 - \lambda + 2\lambda \left(\frac{\theta^2 x_i^2}{2+\theta} + \theta x_i + 1 \right) e^{-\theta x_i} \right). \quad (7)$$

To get the maximum likelihood estimator of λ and θ , we have to solve $\frac{\partial L(\theta, \lambda)}{\partial \theta} = 0$ and $\frac{\partial L(\theta, \lambda)}{\partial \lambda} = 0$. Hence, we have

$$\frac{3n}{\theta} - \frac{n}{2+\theta} - n\bar{x} + 2\lambda \sum_{i=1}^n \frac{\frac{(x_i + 2\theta(\theta + 2)x_i + \theta^2 x_i)e^{-\theta x_i}}{(\theta + 2)^2} - x_i \left(\frac{\theta^2 x_i^2}{\theta + 2} + \theta x_i + 1 \right) e^{-\theta x_i}}{\left(1 - \lambda + 2\lambda \left(\frac{\theta^2 x_i^2}{2+\theta} + \theta x_i + 1 \right) e^{-\theta x_i} \right)} = 0 \quad (8)$$

and

$$\sum_{i=1}^n \left(\frac{2 \left(\frac{\theta^2 x_i^2}{2+\theta} + \theta x_i + 1 \right) e^{-\theta x_i} - 1}{1 - \lambda + 2\lambda \left(\frac{\theta^2 x_i^2}{\theta+2} + \theta x_i + 1 \right) e^{-\theta x_i}} \right) = 0 \quad (9)$$

To get the MLE of λ and θ , we use the R- statistical package to solve these equation numerically.

VIII. Simulation Study

To generate the random sample from transmuted zeghdoudi distribution, we proceed as follows:

Let $F(x) = p$, where p follows *uniform*(0,1).

i.e.

$$(1 + \lambda)G(x) - \lambda G^2(x) = p$$

i.e.

$$U = G(x) = \frac{(1 + \lambda) \pm \sqrt{(1 + \lambda)^2 - 4\lambda p}}{2\lambda}, |\lambda| \leq 1$$

We will choose that solution which lies in (0,1). Here $G(x)$ is the cdf of zeghdoudi random variable. Then, we follow the approach of generating sample from zeghdoudi distribution. Hence the algorithm is

- Generate $p_i \sim \text{uniform}(0,1)$, $i = 1, 2, \dots, n$.
- Calculate $U_i = \frac{(1+\lambda) \pm \sqrt{(1+\lambda)^2 - 4\lambda p_i}}{2\lambda}$, $i = 1, 2, \dots, n$.
- Generate $V_i \sim \text{gamma}(2, \theta)$, $i = 1, 2, \dots, n$.
- Generate $W_i \sim \text{gamma}(3, \theta)$, $i = 1, 2, \dots, n$.
- If $U_i \leq \frac{\theta}{2+\theta}$, then set $X_i = V_i$. Otherwise, set $X_i = W_i$.

A Monte Carlo simulation study is carried out considering $N = 10000$ times for different values of n and presented graphically for $\theta = 0.8$ and $\lambda = 0.6$ (Figure). The following two measures were computed

- Average bias of the simulated estimates $\hat{\theta}_i$, $i = 1, 2, \dots, N$: $\frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)$
- Average Mean Square Error (MSE) of the simulated estimates $\hat{\theta}_i$, $i = 1, 2, \dots, N$: $\frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2$

IX. Data Analysis

In this section, we use a real lifetime data and shown that the flexibility of Transmuted Zeghdoudi Distribution.

Data set 1- represent the waiting times (in mins) before customer service in bank 0.8 0.8 1.3 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5

Table 1: The MLEs of parameter(s), negative log-likelihood and AIC values for two survival model.

Survival model	MLE	negative log-likelihood	AIC -2log-likelihood+2K ,K=parameter
Transmuted Zeghdoudi(θ, λ)	$\hat{\theta} = 0.2593$, $\hat{\lambda} = 0.3784$	314.271	632.542
Zeghdoudi(θ)	$\hat{\theta} = 0.3886$	334.234	670.468
Lindley(θ)	$\hat{\theta} = 0.1865$	319.0468	640.093
Exponential(θ)	$\hat{\theta} = 0.1012$	329.047	660.094
Xgamma(θ)	$\hat{\theta} = 0.2633$	332.237	666.474

Figure 5: The graph showing fitting of the different distribution to the waiting time(in mins) before the customer service.

X. Conclusion

In this article, we introduce a new two parameter Transmuted Zeghdoudi Distribution(TZD) which is an extension of one parameter Zeghdoudi Distribution. We derived some statistical properties, estimation of parameters and application to some real lifetime data sets. An application of the proposed distribution to the “waiting times before customer service in bank” data are given to show that the new distribution provides better fit than other models available in the paper. We expect that this article will serve as a reference for many researchers for future research in this field.

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