

## About Edge Independent Sets in Hypergraphs

D. K. Thakkar<sup>1</sup> and V. R. Dave<sup>2\*</sup>

<sup>1</sup>Department of Mathematics, Saurashtra University, Rajkot – 360 005, India

<sup>2</sup>Shree M. & N. Virani Science College, Kalavad Road, Rajkot – 360 005, India

\*Corresponding Author: varadadave@gmail.com, Tel: +91 9429189787

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**Abstract** - In this paper we have introduced maximum edge independent sets in hypergraphs where a maximum edge independent set is an edge independent set with maximum cardinality. We have proved that the edge independence number of a hypergraph (with minimum edge degree at least two) remain same or decreases when an edge is removed from the hypergraph. In particular we have proved that the edge independence number of a hypergraph (with minimum edge degree at least two and edge independence number is greater than one) decreases if and only if there is a maximum edge independent set which contains the edge. We have also considered the edge independence numbers of the subhypergraph and the partial subhypergraph obtained by removing a vertex from the hypergraph. We observe that edge independence number of the finite projective plane always increases when any vertex is removed from the hypergraph and the subhypergraph is considered and the edge independence number remains same when any vertex is removed and the partial subhypergraph is considered.

**Keywords** - Hypergraph, Edge Independent Set, Maximal Edge Independent Set, Maximum Edge Independent Sets, Edge Independence Number, Subhypergraphs, Partial Subhypergraphs

**AMS Subject Classification (2010):** 05C15, 05C69, 05C65

### I. INTRODUCTION

Domination related parameters for graphs have been studied by several authors [5, 6]. These parameters can also be extended to hypergraphs and new parameters related to hypergraph can be defined [1, 3, 4]. We introduced the concept of edge stable sets & edge independent sets for hypergraphs in [10]. In this paper we introduce the concept of maximum edge independent sets in hypergraphs. We also proved several results considering subhypergraphs and partial subhypergraphs with edge independent sets and maximum edge independent sets.

The present paper begins with the introduction of the concepts studied in the earlier papers. This is followed by preliminaries. The next section consists of main results and some related examples also. The next section contains concluding remarks and future scope of the research. This is followed by acknowledgement. The last section contains the references of this article.

### II. PRELIMINARIES

**Definition 2.1(Hypergraph)** [4] A hypergraph  $G$  is an ordered pair  $(V(G), E(G))$  where  $V(G)$  is a non-empty finite set &  $E(G)$  is a family of non-empty subsets of  $V(G)$   $\ni$  their union  $= V(G)$ . The elements of  $V(G)$  are called *vertices* & the members of  $E(G)$  are called *edges of the hypergraph*  $G$ .

We make the following assumption about the hypergraph.

- (1) Any two distinct edges intersect in at most one vertex.
- (2) If  $e_1$  and  $e_2$  are distinct edges with  $|e_1|, |e_2| > 1$  then  $e_1 \not\subseteq e_2$  &  $e_2 \not\subseteq e_1$

**Definition 2.2(Edge Degree)** [4] Let  $G$  be a hypergraph &  $v \in V(G)$  then the *edge degree* of  $v = d_E(v)$  = the number of edges containing the vertex  $v$ . The minimum edge degree among all the vertices of  $G$  is denoted as  $\delta_E(G)$  and the maximum edge degree is denoted as  $\Delta_E(G)$ .

**Definition 2.3(Dominating Set in Hypergraph)** [1] Let  $G$  be a hypergraph &  $S \subseteq V(G)$  then  $S$  is said to be a *dominating set* of  $G$  if for every  $v \in V(G) - S$  there is  $u \in S \ni u$  and  $v$  are adjacent vertices.

A dominating set with minimum cardinality is called *minimum dominating set* and cardinality of such a set is called *domination number* of  $G$  and it is denoted as  $\gamma(G)$ .

**Definition 2.4(Edge Dominating Set)** [7] Let  $G$  be a hypergraph &  $S \subseteq E(G)$  then  $S$  is said to be an *edge dominating set* of  $G$  if for every  $e \in E(G) - S$  there is some  $f \in S \ni e$  and  $f$  are adjacent edges.

An edge dominating set with minimum cardinality is called a *minimum edge dominating set* and cardinality of such a set is

called *edge domination number* of  $G$  and it is denoted as  $\gamma_E(G)$ .

**Definition 2.5(Sub Hypergraph and Partial Sub Hypergraph)** [3] Let  $G$  be a hypergraph &  $v \in V(G)$ . Consider the subset  $V(G) - \{v\}$  of  $V(G)$ . This set will induce two types of hypergraphs from  $G$ .

(1) First type of hypergraph: Here the vertex set =  $V(G) - \{v\}$  (where  $\{v\}$  is not an edge of  $G$ ) and the edge set =  $\{e' / e' = e - \{v\} \text{ for some } e \in E(G)\}$ . This hypergraph is called the *sub hypergraph* of  $G$  & it is denoted as  $G - \{v\}$ .

(2) Second type of hypergraph: Here also the vertex set =  $V(G) - \{v\}$  and edges in this hypergraph are those edges of  $G$  which do not contain the vertex  $v$ . This hypergraph is called the *partial sub hypergraph* of  $G$ .

**Definition 2.6(Edge independent Set)** [10] Let  $G$  be a hypergraph &  $F$  be a set of edges of  $G$  then  $F$  is said to be an *edge independent set* of  $G$  if no two edges of  $F$  are adjacent.

**Definition 2.7(Maximal Edge Independent Set)** [10] Let  $G$  be a hypergraph and  $F$  be an edge independent set of  $G$  then  $F$  is said to be a *maximal edge independent set* if  $F \cup \{e\}$  is not an edge independent set of  $G$  for every edge  $e \in E(G) - F$ .

### III. MAIN RESULTS

Now we introduce maximum edge independent sets in hypergraphs.

**Definition 3.1(Maximum Edge Independent Set):** Let  $G$  be a hypergraph &  $F$  be an edge independent set of  $G$  then  $F$  is said to be a *maximum edge independent set* if its cardinality is the maximum among all the edge independent sets of  $G$ .

Cardinality of a maximum edge independent set is called *edge independence number* of  $G$  & it is denoted as  $\beta_1(G)$ .

**Example 3.2:** Consider the hypergraph  $G$  whose vertex set  $V(G) = \{1, 2, 3, 4, 5, 6\}$  &  $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

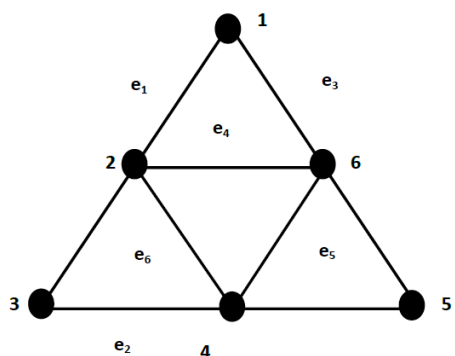


Figure 1

Here,  $F = \{e_2, e_4\}$  is a maximum edge independent set of  $G$ .

$$\therefore \beta_1(G) = 2$$

**Example 3.3:** Consider the finite projective plane with  $r^2 - r + 1$  vertices &  $r^2 - r + 1$  edges. In this hypergraph any two edges have the non-empty intersection.

$\therefore$  A maximum edge independent set will have only one edge in it.

$\therefore$  Every singleton edge set is a maximum edge independent set.

$$\therefore \beta_1(G) = 1$$

**Proposition 3.4:** Let  $G$  be a hypergraph &  $e$  be an edge of  $G$  such that edge degree of each vertex of  $e \geq 2$ . If  $F$  is an edge independent set of  $G - e$  then  $F$  is also an edge independent set of  $G$ .

**Proof:** Let  $F$  be an edge independent set of  $G - e$ . If  $e_1, e_2$  are edges of  $F$  in  $G - e$  then  $e_1, e_2$  are also edges of  $G$  and they are non-adjacent.

$\therefore F$  is an edge independent set of  $G$ . ■

**Theorem 3.5:** Let  $G$  be a hypergraph &  $e$  be an edge of  $G$  such that edge degree of each vertex of  $e \geq 2$  then  $\beta_1(G - e) \leq \beta_1(G)$ .

**Proof:** Let  $F$  be a maximum edge independent set of  $G - e$  then  $F$  is also an edge independent set of  $G$ .

$$\therefore \beta_1(G) \geq |F| = \beta_1(G - e)$$

Thus,  $\beta_1(G - e) \leq \beta_1(G)$ . ■

**Theorem 3.6:** Let  $G$  be a hypergraph &  $e$  be an edge of  $G$  such that edge degree of each vertex of  $e \geq 2$ . Let  $\beta_1(G) > 1$ . Then  $\beta_1(G - e) < \beta_1(G)$  iff for every maximum edge independent set  $F$  of  $G$ ,  $e \in F$ .

**Proof:** Suppose  $\beta_1(G - e) < \beta_1(G)$

Suppose there is a maximum edge independent set  $F$  of  $G \ni e \notin F$  then  $F$  is also an edge independent set of  $G - e$ .

$$\therefore \beta_1(G - e) \geq |F| = \beta_1(G)$$

$\therefore \beta_1(G - e) = \beta_1(G)$  which is a contradiction.

$\therefore e \in F$  for every maximum edge independent set  $F$  of  $G$ .

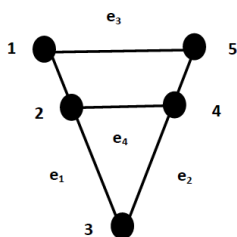
Conversely suppose  $\beta_1(G - e) = \beta_1(G)$

Let  $T$  be a maximum edge independent set of  $G - e$  then  $T$  is also a maximum edge independent set of  $G$  &  $e \notin T$ . which is a contradiction.

$$\therefore \beta_1(G - e) < \beta_1(G)$$

■

**Example 3.7:**



**Figure 2**

Consider the above hypergraph  $G$  whose vertex set  $V(G) = \{1, 2, 3, 4, 5\}$  and edge set  $E(G) = \{e_1, e_2, e_3, e_4\}$

Let  $F = \{e_3, e_4\}$  then  $F$  is a maximum edge independent set of  $G$  & therefore  $\beta_1(G) = 2$

Now, consider hypergraph  $(G - e_4)$ . In this hypergraph  $F_1 = \{e_3\}$  is a maximum edge independent set & therefore  $\beta_1(G - e_4) = 1$ .

$\therefore \beta_1(G - e_4) < \beta_1(G)$

**Subhypergraphs & Edge Independent Sets**

Let  $G$  be a hypergraph &  $v \in V(G)$ . Now, we consider the subhypergraph  $G - v$  whose vertex set is  $V(G) - \{v\}$  & edge set is  $\{e' / e \in E(G)\} (e' = e - \{v\})$

Let  $F$  be a set of edges of  $G$  then  $F'$  will denote the set  $\{e' / e \in F\}$

First we prove the following result.

**Proposition 3.8:** Let  $F$  be an edge independent set of  $G$  then  $F'$  is an edge independent set of  $G - v$ .

**Proof:** Let  $e', f' \in F'$  then  $e, f \in F$  & they are non-adjacent.

Then  $e', f'$  are also non-adjacent.

Thus,  $F'$  is an edge independent set of  $G - v$ . ■

**Theorem 3.9:** Let  $G$  be a hypergraph &  $v \in V(G)$  then  $\beta_1(G - v) \geq \beta_1(G)$ .

**Proof:** Let  $F$  be a maximum edge independent set of  $G$ . Then  $F'$  is an edge independent set of  $G - v$ .

$\therefore \beta_1(G - v) \geq |F'| = |F| = \beta_1(G)$

$\therefore \beta_1(G - v) \geq \beta_1(G)$ . ■

Now, we state & prove a necessary & sufficient condition under which the edge independence number of a hypergraph increases when a vertex is removed from the hypergraph.

**Theorem 3.10:** Let  $G$  be a hypergraph &  $v \in V(G)$  then  $\beta_1(G - v) > \beta_1(G)$  iff for every maximum edge independent set  $F'$  of  $G - v$  there are edges  $e' & f'$  in  $F'$   $\exists e \cap f = \{v\}$

**Proof:** First suppose that  $\beta_1(G - v) > \beta_1(G)$ . Let  $F'$  be any maximum edge independent set of  $G - v$ . Since  $|F| = |F'| > \beta_1(G)$ ,  $F$  cannot be an edge independent set of  $G$ .

$\therefore$  There are edges  $e & f$  in  $F \exists e \cap f \neq \emptyset$  but  $e' & f'$  are edges of  $F'$  &  $F'$  is an edge independent set of  $G - v$ .

$\therefore e' \cap f' = \emptyset$

It follows that  $e \cap f = \{v\}$

Conversely suppose that the condition is satisfied.

Suppose  $\beta_1(G - v) = \beta_1(G)$ .

Let  $F$  be a maximum edge independent set of  $G$ . Since  $|F| = |F'|$ ,  $F'$  is a maximum edge independent set of  $G - v$ .

Let  $e', f' \in F'$  then  $e, f \in F$ .

Since  $F$  is an edge independent set of  $G$ ,  $e \cap f = \emptyset$ . This is a contradiction.

Thus,  $\beta_1(G - v) = \beta_1(G)$  is not possible.

$\therefore \beta_1(G - v) > \beta_1(G)$  ■

**Example 3.11:** Consider the finite projective plane  $G$  with  $r^2 - r + 1$  vertices &  $r^2 - r + 1$  edges ( $r \geq 2$ ). Since any two edges have the non-empty intersection the edge independence number of this hypergraph is 1.

Let  $v \in V(G)$ . Let  $F$  = the set of all edges which contains the vertex  $v$ . Therefore,  $|F| = r$ . Now, consider  $F' = \{e' \exists e \in F\}$  then  $|F'| = r$ . Note that  $F'$  is an edge set of the subhypergraph  $G - v$  & any two edges in  $F'$  are non-adjacent in  $G - v$ . Therefore,  $F'$  is an edge independent subset of  $G - v$ .

$\therefore \beta_1(G - v) \geq |F'| = |F| = r > 1 = \beta_1(G)$ .

Thus,  $\beta_1(G - v) > \beta_1(G)$ .

Suppose  $\beta_1(G - v) > r$ . Let  $T$  be a maximum edge independent subset of  $G - v$  then  $|T| > r$ .

Let  $h' \in T \exists h' \notin F'$ . Let  $e' \in F'$ . Since  $e \cap h \neq \emptyset$  &  $e \cap h \neq \{v\}$ ,  $e \cap h = \{w\}$  for some  $w \neq v$ .  $\therefore e' \cap h' = \{w\}$ . This contradicts the fact that  $T$  is an edge independent subset of  $G - v$ .

$\therefore \beta_1(G - v) > r$  is not possible.

$\therefore \beta_1(G - v) = r$ .

**Partial Subhypergraphs & Edge Independent Set**

Let  $G$  be a hypergraph &  $v \in V(G)$ . Now, we consider the partial subhypergraph  $G - v$  whose vertex set is  $V(G) - \{v\}$  & edges are those edges of  $G$  which do not contain the vertex  $v$ .

**Proposition 3.12:** Let  $G$  be a hypergraph &  $v \in V(G)$ . If  $F$  is an edge independent set of  $G - v$  then it is edge independent subset of  $G$ .

**Proof:** Let  $e_1$  &  $e_2$  be two edges of  $F$ . Since  $F$  is an edge independent set of  $G - v$ ,  $e_1$  &  $e_2$  are non - adjacent. Since  $e_1$  &  $e_2$  are edges of  $G$  also they are non - adjacent edges of  $G$ .

$\therefore F$  is an edge independent subset of  $G$ . ■

**Theorem 3.13:** Let  $G$  be a hypergraph &  $v \in V(G)$  then  $\beta_1(G - v) \leq \beta_1(G)$ .

**Proof:** Let  $F$  be a maximum edge independent set of  $G - v$ . By the above proposition  $F$  is an edge independent subset of  $G$ .

$\therefore \beta_1(G) \geq |F| \geq \beta_1(G - v)$

$\therefore \beta_1(G - v) \leq \beta_1(G)$ . ■

Now, we state & prove a necessary & sufficient condition under which the edge independence number of a hypergraph remains same when a vertex is removed from the hypergraph.

**Theorem 3.14:** Let  $G$  be a hypergraph &  $v \in V(G)$  then  $\beta_1(G - v) = \beta_1(G)$  iff there is a maximum edge independent subset  $F$  of  $G \ni F$  does not contain any edge containing  $v$ .

**Proof:** First suppose that  $\beta_1(G - v) = \beta_1(G)$ . Let  $F$  be a maximum edge independent subset of  $G - v$ . Since  $\beta_1(G - v) = \beta_1(G)$ ,  $F$  is also a maximum edge independent subset of  $G$ . Obviously  $F$  does not contain any edge containing  $v$ .

Conversely suppose there is a maximum edge independent subset  $F$  of  $G \ni F$  does not contain any edge containing  $v$  then  $F$  is a set of edges of  $G - v$  & it is also an edge independent subset of  $G - v$ .

$\therefore \beta_1(G - v) \geq |F| = \beta_1(G) \geq \beta_1(G - v)$

$\therefore \beta_1(G - v) = \beta_1(G)$  ■

**Corollary 3.15:** Let  $G$  be a hypergraph &  $v \in V(G)$  then  $\beta_1(G - v) < \beta_1(G)$  iff for every maximum edge independent subset  $F$  of  $G \exists$  an edge  $e \in F \ni e$  contains the vertex  $v$ .

**Proof:** Follows from the above theorem.

**Example 3.16:** Consider the hypergraph  $G$  whose vertex set  $V(G) = \{1, 2, 3, 4, 5, 6\}$  &  $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

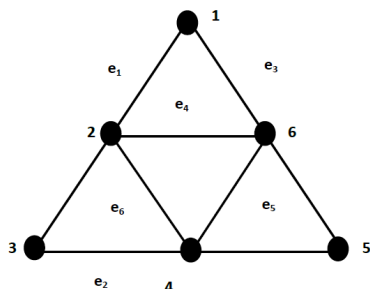


Figure 3

Here,  $F = \{e_2, e_4\}$  is a maximum edge independent set of  $G$ .

$\therefore \beta_1(G) = 2$

Now, consider the partial subhypergraph  $G - \{1\}$

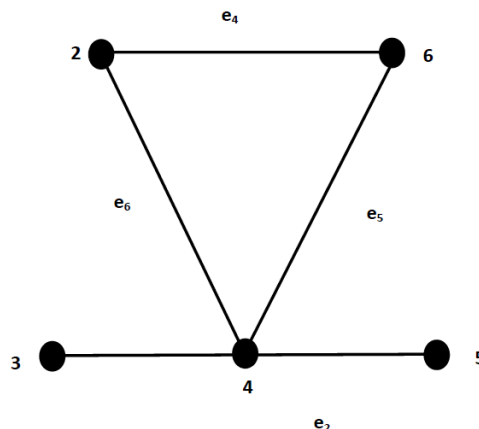


Figure 4

Then,  $F = \{e_2, e_4\}$  is also a maximum edge independent set of  $G - \{1\}$ .

$\therefore \beta_1(G - \{1\}) = 2$

$\therefore \beta_1(G - v) = \beta_1(G)$

Now, consider the partial subhypergraph  $G - \{2\}$

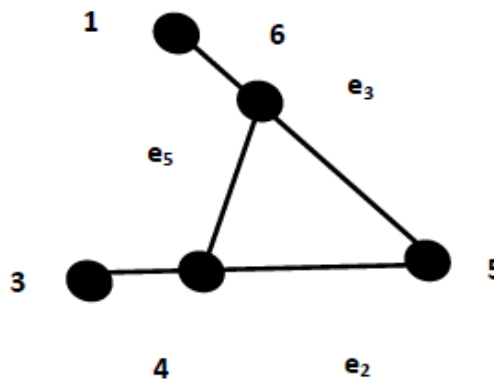


Figure 5

Then  $F_1 = \{e_5\}$  is a maximum edge independent set of  $G - \{2\}$ .

$\therefore \beta_1(G - \{4\}) = 1$

$\therefore \beta_1(G - v) < \beta_1(G)$

**Example 3.17:** Consider the finite projective plane  $G$  with  $r^2 - r + 1$  vertices &  $r^2 - r + 1$  edges ( $r \geq 2$ ). Since any two edges have the non-empty intersection the edge independence number of this hypergraph is 1.

If  $v \in V(G)$  then there are exactly  $r$  edges containing  $v$ .

$\therefore$  There is an edge  $e \ni v \notin e$ . Then  $\{e\}$  is a maximum edge independent set of  $G \ni$  any edge of it does not contain the vertex  $v$ .

$\therefore \beta_1(G - v) = \beta_1(G)$ .

#### IV. CONCLUSION AND FUTURE SCOPE

It may happen that the edge independence number remains same when some edges are removed from the hypergraph. It may be interesting to investigate and to find the lower bound of minimum number of edges whose removal decreases the edge independence number of a given hypergraph. It may be interesting to find the relation between the lower bound of the hypergraph  $G$  and the lower bound for the subhypergraph or partial subhypergraph obtained by removing a vertex from the hypergraph.

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#### AUTHORS PROFILE

**D. K. Thakkar** is a retired professor of the Department of Mathematics of Saurashtra University, Rajkot. His areas of interest are Graph Theory, Topology and Discrete Mathematics. He has published over 80 research papers in various journals.



**V. R. Dave** is a young research student who likes to work in a challenging environment. She is working as an Assistant Professor in Shree M. and N. virani Science College. Her area of interest is Graph Theory.

